

On a conjecture for an overdetermined problem for the biharmonic operator

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Received 28 March 2007; accepted 28 March 2007

Abstract

It has been proven that if the solution exists to an inhomogeneous biharmonic equation in the plane where the values of the solution, the normal derivative of the solution, and the Laplacian of the solution are prescribed on the boundary, then the domain is a disk. This result has been extended to N -dimensions by the Serrin reflection method. Here we present a new proof and give a characterization of open balls in $\mathbb{R}^n$.

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Keywords: Biharmonic operator; Overdetermined problem; Spherical symmetry

1. Introduction

In response to a fluid flow question posed by a colleague, Serrin [1] proved that the solution to Poisson's equation for which the values of the solution and the outward normal derivative are specified on the boundary is spherically symmetric. In particular, Serrin proved that if u satisfies the overdetermined problem

$$\Delta u = -1 \quad \text{in } D, \quad u = 0, \quad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial D, \quad (1.1)$$

where D is a bounded smooth domain in $\mathbb{R}^N$, Δ is the N -dimensional Laplace operator, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative operator, then D is an N -ball of radius a and $u = (a^2 - r^2)/2N$, where r is the distance from the center of the ball. This result was also extended to the other more general elliptic equations and somewhat different boundary conditions in [1].

Many authors have extended Serrin's result to overdetermined problems for second-order and higher-order elliptic equations and systems in interior or exterior domains as well as for ring-shaped domains. In these works, the necessity of the spherical nature of the boundary was determined. A brief survey of some of these extensions and the methods used is presented in [2].

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In [2] the authors considered several fourth-order overdetermined boundary value problems for the biharmonic operator, $\Delta^2 = \Delta\Delta$, which have applications in $\mathbb{R}^2$ to problems in linear isotropic elasticity and slow flows of viscous fluids. All of the results in [2] were valid in N -space, $N \geq 2$, with the exception of the following theorem.

Theorem 1.1. *If u satisfies*

$$\Delta^2 u = 1 \quad \text{in } D, \quad u = \Delta u = 0, \quad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial D, \quad (1.2)$$

for $D \subset \mathbb{R}^2$, where D is star-shaped with respect to the origin and $\partial D \in C^{2+\varepsilon}$, then D is a disk.

It was conjectured that this result holds for $D \subset \mathbb{R}^N$, $N > 2$, and was proved in [3] by the Serrin reflection method. We present a new proof by means of suitably defined auxiliary functions in Section 2.

2. Proof of the conjecture

We consider the boundary value problem

$$\Delta^2 u = 1 \quad \text{in } D, \quad u = \Delta u = 0, \quad \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial D, \quad (2.1)$$

where D is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and c is a constant. An equivalent second-order problem is

$$\Delta u = -\psi \quad \text{in } \overline{D}, \quad u = 0, \quad \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial D, \quad (2.2)$$

where ψ is the torsion function, i.e. ψ satisfies

$$\Delta \psi = -1 \quad \text{in } D, \quad \psi = 0 \quad \text{on } \partial D. \quad (2.3)$$

It follows from (2.2) that

$$c = -\frac{1}{S} \int_D \psi \, dx, \quad (2.4)$$

where S is the surface area of the boundary ∂D .

We define the auxiliary function v by

$$v = \frac{N+2}{2N} u - \frac{\psi^2}{4}, \quad (2.5)$$

and note that v satisfies the overdetermined problem

$$\Delta v = -\left(\frac{1}{2} \psi_{,i} \psi_{,i} + \frac{\psi}{N}\right) \quad \text{in } \overline{D}, \quad v = 0, \quad \frac{\partial v}{\partial \nu} = \frac{N+2}{2N} c \quad \text{on } \partial D, \quad (2.6)$$

where we use the comma convention for partial differentiation and the summation convention on repeated indices.

We now consider the function w defined by

$$w = v_{,i} v_{,i} - v \Delta v - \frac{N-2}{N} M^2 \psi, \quad (2.7)$$

where

$$M^2 = \max_{\overline{D}} (\Delta v)^2,$$

and note that its Laplacian can be written as

$$\Delta w = 2 \left[v_{,ij} v_{,ij} - \frac{1}{N} (\Delta v)^2 \right] + \frac{N-2}{N} \left[M^2 - (\Delta v)^2 \right] + v \left[\psi_{,ij} \psi_{,ij} - \frac{1}{N} (\Delta \psi)^2 \right].$$

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