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On the Chebyshev type inequality for seminormed fuzzy integral

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ABSTRACT

The Chebyshev type inequality for seminormed fuzzy integral is discussed. The main results of this paper generalize some previous results obtained by the authors. We also investigate the properties of semiconormed fuzzy integral, and a related inequality for this type of integral is obtained.

1. Introduction

Since they were introduced by Sugeno [1], fuzzy measures and fuzzy integrals have been intensively studied (see, e.g., [2–7]). Many authors generalized the Sugeno integral by using some other operators to replace the special operator(s) $\wedge$ and/or $\vee$ (see, e.g., [8–15]). In [11], Suárez and Gil presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

The Chebyshev type inequality [16] for the Sugeno integral was initiated by Flores-Franulič and Román-Flores [17], and followed by Ouyang et al. [18,19]. In [17] Flores-Franulič and Román-Flores proved a Chebyshev type inequality for the Lebesgue measure-based Sugeno integral and for two continuous and strictly monotone functions (in the same sense). Based upon some results of [20], Ouyang, Fang and Wang [19] generalized the results of [17]. They proved the Chebyshev type inequality for the arbitrarily fuzzy measure-based (the universal space is a subset of the real space R) Sugeno integral and for two monotone functions (in the same sense). Later on, Mesiar and Ouyang [18] proved a Chebyshev inequality for comonotone functions and for an abstract operator $\star$. Observe that comonotone functions can be defined in any space. Thus Mesiar and Ouyang proved the fuzzy Chebyshev inequality in an abstract space.

The main aim of the present work is to prove a Chebyshev type inequality for the seminormed fuzzy integrals. After some preliminaries and summarization of some previous known results in Section 2, Section 3 deals with a general Chebyshev type inequality for seminormed fuzzy integrals. Section 4 includes a reverse inequality for semiconormed fuzzy integrals. Finally, some conclusions are given.

2. Preliminaries

In this section we recall some basic definitions and previous results which will be used in what follows.

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Let $\mathbf{N}$ denote the set of all positive integers. Throughout this paper, all considered subsets are supposed to belong to $\mathcal{F}$.

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Definition 2.1 (*Sugeno [1]*). A set function $\mu : \mathcal{F} \to [0, 1]$ is called a fuzzy measure if the following properties are satisfied:
(FM1) $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
(FM2) $A \subset B$ implies $\mu(A) \leq \mu(B)$;
(FM3) $A_n \to A$ implies $\mu(A_n) \to \mu(A)$.
When μ is a fuzzy measure, the triple $(X, \mathcal{F}, \mu)$ is called a fuzzy measure space.

Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space,and $\mathcal{F}_+(X) = \{f | f : X \to [0, 1] \text{ is measurable with respect to } \mathcal{F}\}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. For any $\alpha \in [0, 1]$, we will denote the set $\{x \in X | f(x) \geq \alpha\}$ by F_α and $\{x \in X | f(x) > \alpha\}$ by $F_{\bar{\alpha}}$. Clearly, both F_α and $F_{\bar{\alpha}}$ are nonincreasing with respect to α, i.e., $\alpha \leq \beta$ implies $F_\alpha \supseteq F_\beta$ and $F_{\bar{\alpha}} \supseteq F_{\bar{\beta}}$.

Definition 2.2 (*Pap [3], Sugeno [1]*). Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space and $A \in \mathcal{F}$, the Sugeno integral of f over A, with respect to the fuzzy measure μ, is defined by

$$(S) \int_A f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)) .$$

When $A = X$, then

$$(S) \int_X f \mathrm{d}\mu = (S) \int f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha)) .$$

Notice that Ralescu and Adams (see [5]) extended the range of fuzzy measures and the Sugeno integrals from [0, 1] to $[0, \infty]$. But in this paper, we only deal with the original fuzzy measures and the Sugeno integrals which was introduced by Sugeno in 1974.

Note in the above definition, $\wedge$ is just the prototypical t-norm minimum and $\vee$ the prototypical t-conorm maximum. A t-norm [21] is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:
(A) $T(x, 1) = T(1, x) = x \; \forall x \in [0, 1]$.
(B) $\forall x_1, x_2, y_1, y_2$ in [0, 1], if $x_1 \leq x_2, y_1 \leq y_2$ then $T(x_1, y_1) \leq T(x_2, y_2)$.
(C) $T(x, y) = T(y, x)$.
(D) $T(T(x, y), z) = T(x, T(y, z))$.
A function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm [21], if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$. Evidently, a t-conorm S satisfies:
(A′) $S(x, 0) = S(0, x) = x, \; \forall x \in [0, 1]$ as well as conditions (B), (C) and (D).
A binary operator T (S) on [0, 1] is called a t-seminorm (t-semiconorm) [11] if it satisfies the above conditions (A) and (B) ((A′) and (B)). Notice that in the literature, a t-seminorm is also called a semicopula [22]. By using the concepts of t-seminorm and t-semiconorm, Suárez and Gil proposed two families of fuzzy integrals:

Definition 2.3. Let T be a t-seminorm, then the seminormed fuzzy integral of f over A with respect to T and the fuzzy measure μ is defined by

$$\int_{T,\, A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T\left(\alpha, \mu(A \cap F_\alpha)\right) . \tag{2.1}$$

Definition 2.4. Let S be a t-semiconorm, then the semiconormed fuzzy integral of f over A with respect to S and the fuzzy measure μ is defined by

$$\int_{S,\, A} f \mathrm{d}\mu = \bigwedge_{\alpha \in [0,1]} S(\alpha, \mu(A \cap F_{\bar{\alpha}})).$$

Remark 2.5. Maybe one can define the so-called conormed-seminormed fuzzy integral $\int_{TS,\, A} f \mathrm{d}\mu$ by using an arbitrary t-conorm S to replace the special t-conorm "$\bigvee$" in (2.1). However, as Suárez and Gil [11] pointed out that the conormed-seminormed integral satisfies the essential property

$$\int_{TS,\, X} a \mathrm{d}\mu = a, \quad \forall a \in [0, 1]$$

if and only if $S = \bigvee$. The case of semiconormed fuzzy integral is similar.