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Existence of maximal and minimal periodic solutions for first-order functional differential equations[☆]

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ABSTRACT

One important question in population models is whether periodic solutions exist and whether they are bounded between minimal and maximal solutions. This paper deals with the existence of maximal and minimal periodic solutions for the periodic solutions of a first-order functional differential equation

$$y'(t) = -a(t)y(t) + f(t, y(t - \tau(t)))$$

by using the method of lower and upper solutions.

Functional differential equations with periodic delays appear in a number of ecological, economical, control and physiological models. One important question is whether these equations can support periodic solutions. Such questions have been studied extensively by a number of authors (see for example, [1–5] and the references therein).

In general, if there exist a priori bounds for periodic solutions, we can use them in fixed point theorems for locating the desired solutions. A natural question then arises as to whether there exist more general bounding functions such as lower and upper periodic solutions, maximal and minimal periodic solutions, etc.

This paper deals with the above problem for periodic solutions of the first-order functional differential equations

$$y'(t) = -a(t)y(t) + f(t, y(t - \tau(t))). \tag{1}$$

In what follows, we will assume that $a = a(t)$ and $\tau = \tau(t)$ are continuous T-periodic functions. We also assume that $T > 0$, that $f \in C\left(R^2, R\right)$ is a continuous function and T-periodic with respect to the first variable and nondecreasing with respect to the second variable, and that $a(t) > 0$ for $t \in R$.

In the remainder of this section, we provide some background definitions.

Definition 1. Let E and F be ordered Banach spaces with ordering $\leq$. Let $D \subset E$. An operator A is called an increasing operator on D if $A : D \to F$ and $Ax \leq Ay$ for any $x, y \in D$ and $x \leq y$.

Definition 2. Let E be an ordered Banach space. Let $D \subset E$ and $A : D \to E$. $x_0 \in D$ is said to be a lower solution of the operator equation $x = Ax$ if $x_0 \leq Ax_0$. We also call the point x_0 a lower solution of A. $y_0 \in D$ is said to be an upper solution of the operator equation $x = Ax$ if $Ay_0 \leq y_0$. We also call the point y_0 an upper solution of A.

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The proof of the following theorem can be found in [6].

Theorem 1. *Let E be an ordered Banach space, and let $x_0, y_0 \in E$ such that $x_0 \leq y_0$. Let $D = [x_0, y_0]$ be the set of $u \in E$ such that $x_0 \leq u \leq y_0$ and $A : D \to E$. Suppose that*

(1) *A is an increasing operator;*
(2) *x_0 is a lower solutions of A, y_0 is an upper solutions of A;*
(3) *$A \in C(D, E)$;*
(4) *$A(D)$ is a precompact set of E.*

Then

(1) A has a minimal fixed point x^* and maximal fixed point y^*, i.e. $x^* = Ax^*$, $y^* = Ay^*$, and any fixed point z^* of A in $[x_0, y_0]$ belongs to $[x^*, y^*]$.

(2) Set

$$x_n = Ax_{n-1}, \quad y_n = Ay_{n-1}, \quad n \in N. \tag{2}$$

Then

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0 \tag{3}$$

and

$$x_n \to x^*, \quad y_n \to y^*, \quad n \to \infty. \tag{4}$$

It has been shown that the Eq. (1) has a T-periodic solution $y(t)$ if, and only if $y(t)$ is a T-periodic solution of the equation

$$y(t) = \int_t^{t+T} G(t, s) f(s, y(s - \tau(s)))\mathrm{d}s, \tag{5}$$

where

$$G(t, s) = \frac{\exp\left(\int_t^s a(u)\mathrm{d}u\right)}{\exp(\int_0^T a(u)\mathrm{d}u) - 1}$$

(see e.g. [1,2]). Therefore, we may transform our existence problem into a fixed point problem. To this end, we first note that

$$0 < m \equiv \min_{0 \leq t, s \leq T} G(t, s) \leq G(t, s) \leq \max_{0 \leq t, s \leq T} G(t, s) \equiv M < \infty.$$

The existence of periodic solutions for the Eq. (5) has been studied extensively by a number of authors when $\tau \equiv 0$. (See [1–5] and the references therein).

Now let $C_T(R)$ be the set of all real T-periodic continuous functions defined on R which is endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{t \in [0,T]} |y(t)|.$$

Set $P_0 = \{\phi \in C_T(R) : \phi(x) \geq 0, x \in R\}$. Then it is easy to see that $P_0 \subset C_T(R)$ is a normal cone, and P_0 induces an ordering in E given by $x \leq y$, if and only if $y - x \in P_0$.

Definition 3. $v_0 \in C_T^1(R)$ is called a lower solution of Eq. (1) if it satisfies the following condition

$$v_0'(t) \leq -a(t)v_0(t) + f(t, v_0(t - \tau(t))), \tag{6}$$

similarly, we say $\omega_0 \in C_T^1(R)$ is a upper solution of Eq. (1) if it satisfies the following condition

$$\omega_0'(t) \geq -a(t)\omega_0(t) + f(t, \omega_0(t - \tau(t))). \tag{7}$$

Theorem 2. *Suppose that $v_0(t)$ and $\omega_0(t)$ are respectively lower and upper solutions for Eq. (1), and $v_0(t) \leq \omega_0(t)$. Then Eq. (1) has a minimal solution $v^*(t)$ and a maximal solution $\omega^*(t)$ in $D = \{u \in C_T(R) : v_0 \leq u \leq \omega_0\}$. Set*

$$v_n(t) = \int_t^{t+T} G(t, s) f(s, v_{n-1}(s - \tau(s)))\mathrm{d}s \quad n \in N,$$

and

$$\omega_n(t) = \int_t^{t+T} G(t, s) f(s, \omega_{n-1}(s - \tau(s)))\mathrm{d}s \quad n \in N.$$

Then $\{v_n(t)\}$, $\{\omega_n(t)\}$ are monotonically and uniformly convergent to $v^*(t)$ and $\omega^*(t)$, respectively, and any periodic solution $y(t)$ of Eq. (1) in $[v_0, w_0]$ belongs to $[v^*, \omega^*]$.