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Remark on the regularity criterion for three-dimensional magnetohydrodynamic equations

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ABSTRACT

In this letter, we consider the three-dimensional magnetohydrodynamic (3D MHD) equations and we prove a new regularity criterion for weak solutions. More precisely, we show that if the velocity field $u \in L^{\frac{2}{1-r}}\left(0, T, \dot{X}_r\left(\mathbb{R}^3\right)\right)$ with $r \in [0, 1[$, then the solution remains smooth on $[0, T]$. Since

$$L^{\frac{3}{r}}\left(\mathbb{R}^3\right) \subset \dot{X}_r\left(\mathbb{R}^3\right)$$

for $0 \leq r < \frac{3}{2}$, our result improve the results in He and Xin (2005) [1] and Zhou (2005) [2].

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1. Introduction

In this letter, we consider the initial value problem for the three-dimensional magnetohydrodynamic (3D MHD) equations in $\mathbb{R}^3 \times (0, T)$:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p + \frac{1}{2}\nabla |b|^2 - b \cdot \nabla b = 0 \\ \partial_t b - \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0 \\ \nabla \cdot u = \nabla \cdot b = 0 \\ u(x, 0) = u_0(x), \qquad b(x, 0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = u(x, t)$ is the velocity field, $b \in \mathbb{R}^3$ is the magnetic field, and $p = p(x, t)$ is the scalar pressure, while u_0 and b_0 are the given initial velocity and initial magnetic field with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution. For simplicity, we assume that the external force has a scalar potential and is included in the pressure gradient.

It is well known [3] that problem (1.1) is locally well posed for any given initial datum $u_0, b_0 \in H^s\left(\mathbb{R}^3\right)$, $s \geq 3$. Moreover, just as for other mechanical equations, say Navier–Stokes equations, it has been proved by Wu [4] that (1.1) has a weak solution for any given $u_0, b_0 \in L^2\left(\mathbb{R}^3\right)$, with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. But whether the unique local solution can exist globally or the weak solution is regular and unique is a challenging outstanding problem: just as for the 3D Navier–Stokes equations, the regularity of the weak solution for the 3D MHD equations remains open [3].

Recently, He and Xin [1] and Zhou [2] proved that if the velocity field $u \in L^s\left(0, T, L^r\left(\mathbb{R}^3\right)\right)$ with $\frac{2}{s} + \frac{3}{r} = 1$ and $r > 1$, then the solution (u, b) remains smooth on $[0, T]$. In particular, they established the Serrin-type criterion in terms of the velocity field without imposing any restriction on the magnetic field b (see [2]).

The purpose of this letter is to improve and extend some known regularity criterion of weak solution for the MHD equations in multiplier spaces. More precisely, we prove if the velocity field

$$u \in L^{\frac{2}{1-r}}\left(0, T, \dot{X}_r\left(\mathbb{R}^3\right)\right) \quad \text{for } 0 \leq r < 1,$$

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then the solution remains smooth on $[0, T]$. Since

$$L^{\frac{3}{r}}\left(\mathbb{R}^3\right) \subset \dot{X}_r\left(\mathbb{R}^3\right)$$

for $0 \le r < \frac{3}{2}$, our result improves the results in [1,2].

2. Multiplier spaces

In this section, we recall the definition and some properties of the space we are going to use. These spaces play an important role in studying the regularity of solutions to partial differential equations; see e.g. [5] and references therein.

Definition 2.1. For $0 \le r < \frac{3}{2}$, the space $\dot{X}_r$ is defined as the space of $f(x) \in L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2} < \infty,$$

where we denote by $\dot{H}^r\left(\mathbb{R}^3\right)$ the completion of the space $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm $\|u\|_{\dot{H}^r} = \left\|(-\Delta)^{\frac{r}{2}}u\right\|_{L^2}$.

We have the homogeneity properties:

$$\|f(.+x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}, \quad \forall x_0 \in \mathbb{R}^3,$$

$$\|f(\lambda .)\|_{\dot{X}_r} = \frac{1}{\lambda^r}\|f\|_{\dot{X}_r}, \quad \forall \lambda > 0.$$

Example 1. Due to the well-known inequality

$$\left\|\frac{g}{|x|}\right\|_{L^2} \le 2\|\nabla g\|_{L^2},$$

we see that $|x|^{-1} \in \dot{X}_1\left(\mathbb{R}^3\right)$.

Additionally, we have the following inclusion:

Lemma 1. $L^{\frac{3}{r}}\left(\mathbb{R}^3\right) \subset \dot{X}_r\left(\mathbb{R}^3\right)$, $0 \le r < \frac{3}{2}$ *with continuous injection.*

Proof. Let $f \in L^{\frac{3}{r}}\left(\mathbb{R}^3\right)$. By using the following well-known Sobolev embedding:

$$\dot{H}^r\left(\mathbb{R}^3\right) \subset L^q\left(\mathbb{R}^3\right)$$

with $\frac{1}{q} = \frac{1}{2} - \frac{r}{3}$, we have, by Hölder' s inequality,

$$\begin{aligned}\|fg\|_{L^2} &\le \|f\|_{L^{\frac{3}{r}}}\|g\|_{L^q} \\ &\le \|f\|_{L^{\frac{3}{r}}}\|g\|_{\dot{H}^r}.\end{aligned}$$

Then, it follows that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1}\|fg\|_{L^2} \le C\|f\|_{L^{\frac{3}{r}}}. \quad \blacksquare$$

3. Regularity theorem

In this section we give the regularity criterion by velocity to the Leray-type weak solution of the MHD equation (1.1). Before turning our attention to regularity issues, we start with some prerequisites for our main result. Let

$$C_{0,\sigma}^\infty\left(\mathbb{R}^3\right) = \left\{\varphi \in \left(C_0^\infty\left(\mathbb{R}^3\right)\right)^3 : \operatorname{div}\varphi = 0\right\} \subseteq \left(C_0^\infty\left(\mathbb{R}^3\right)\right)^3.$$

The subspace

$$L^2_\sigma\left(\mathbb{R}^3\right) = \overline{C_{0,\sigma}^\infty\left(\mathbb{R}^3\right)}^{\|.\|_{L^2}} = \left\{u \in L^2\left(\mathbb{R}^3\right)^3 : \operatorname{div} u = 0\right\}$$

is obtained as the closure of $C_{0,\sigma}^\infty$ with respect to L^2-norm $\|.\|_{L^2}$. H^r_σ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the norm

$$\|u\|_{H^r} = \|u\|_{L^2} + \left\|(1-\Delta)^{\frac{r}{2}}u\right\|_{L^2}, \quad \text{for } r \ge 0.$$

Our result is the following: