Applied Mathematics Letters 22 (2009) 1534–1538

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Uzawa block relaxation domain decomposition method for a two-body frictionless contact problem

Jonas Koko

LIMOS, Université Blaise Pascal, CNRS UMR 6158, Campus des Cézeaux, BP 10125, 63173 Aubiére cedex, France

ARTICLE INFO

Article history:
Received 27 September 2007
Received in revised form 19 January 2009
Accepted 30 March 2009

Keywords:
Two-body frictionless contact
Augmented Lagrangian
Kuhn–Tucker conditions
Uzawa methods
Domain decomposition

ABSTRACT

We propose a Uzawa block relaxation domain decomposition method for a two-body frictionless contact problem. We introduce auxiliary variables to separate subdomains representing linear elastic bodies. Applying a Uzawa block relaxation algorithm to the corresponding augmented Lagrangian functional yields a domain decomposition algorithm in which we have to solve two uncoupled linear elasticity subproblems in each iteration while the auxiliary variables are computed explicitly using Kuhn–Tucker optimality conditions.

1. Introduction

Contact between elastic bodies is of great interest in engineering (rails, gears, forming, etc.) and poses a serious challenge, for numerical simulation, due to contact laws which require a detailed geometrical description. We propose, in this paper, a domain decomposition method based on augmented Lagrangian in which the interface is the contact surface. The method consists in solving, iteratively, two uncoupled linear elasticity subproblems.

We consider two elastic bodies, each of them occupying a open bounded domain $\Omega^\alpha \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\Gamma^\alpha, \alpha = 1, 2$. We assume that $\Gamma^\alpha = \Gamma_D^\alpha \cup \Gamma_N^\alpha \cup \Gamma_c^\alpha$ where $\{\Gamma_D^\alpha, \Gamma_N^\alpha, \Gamma_c^\alpha\}$ is a partition of Γ^α with $\mathrm{mes}(\Gamma_D^\alpha) > 0$ and $\mathrm{mes}(\Gamma_c^\alpha) > 0$. On Γ_D^α a displacement is prescribed while on Γ_N^α a surface traction is prescribed. Γ_c^α is a part of Γ^α where both bodies may come in contact in the deformed configuration.

Let $\boldsymbol{u}^\alpha$ be the displacement fields of the body Ω^α ($\boldsymbol{u}^\alpha(x) \in \mathbb{R}^2$). We set $\mathbf{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$ as the displacement field of the two-body system. Hooke's law is assumed for each elastic body, i.e. the constitutive equations are

$$\epsilon(\mathbf{v}^\alpha) = \frac{1}{2}(\nabla \mathbf{v}^\alpha + (\nabla \mathbf{v}^\alpha)^t), \qquad \sigma^\alpha(\mathbf{v}^\alpha) = \mu_\alpha \epsilon(\mathbf{v}^\alpha) + \lambda_\alpha \operatorname{tr}(\epsilon(\mathbf{v}^\alpha))\mathbb{I}_d,$$

where $\lambda_\alpha \geq 0$ and $\mu_\alpha > 0$ denote the Lamé constants and $\mathbb{I}_d$ the 2×2 identity matrix. The equilibrium equations are

$$-\operatorname{div} \sigma^\alpha(\boldsymbol{u}^\alpha) = \mathbf{f}^\alpha \quad \text{in } \Omega^\alpha, \tag{1.1}$$

$$\sigma^\alpha(\boldsymbol{u}^\alpha) \cdot \mathbf{n}^\alpha = \mathbf{g}^\alpha \quad \text{on } \Gamma_N^\alpha, \quad \boldsymbol{u}^\alpha = 0 \quad \text{on } \Gamma_D^\alpha, \tag{1.2}$$

where $\mathbf{n}^\alpha$ stands for the unit outward normal to Ω^α.

For a complete formulation, it remains to introduce the set of admissible displacement fields. We can identify both surfaces Γ_c^α by their projection Γ_c on a suitable straight line so that the kinematical contact condition can be described by

$$(\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot \mathbf{n} - g \leq 0 \quad \text{on } \Gamma_c, \tag{1.3}$$

E-mail addresses: koko@isima.fr, koko@sp.isima.fr.

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.03.021

where $\mathbf{n}$ is the unit outward normal to Γ_c (from Ω^1 side) and g the normal gap. We refer, e.g., to [1, Section 6.8] for a derivation of Γ_c from the parametric definition of Γ_c^α. For simplicity, we set $[\mathbf{u}_n] = (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot \mathbf{n}$, the relative normal displacement on the contact interface Γ_c.

2. Augmented Lagrangian formulation

Let us introduce the Hilbert spaces of virtual displacements

$$V^\alpha = \left\{\mathbf{v} \in (H^1(\Omega^\alpha))^2 \;:\; \mathbf{v} = 0 \text{ on } \Gamma_D^\alpha\right\}, \quad \alpha = 1, 2.$$

We set $\mathbf{V} = V^1 \times V^2$ and, for the set of kinematically admissible displacement fields,

$$K = \left\{\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in \mathbf{V},\; [\mathbf{v}_n] - g \le 0 \text{ on } \Gamma_c\right\}.$$

For $\boldsymbol{u}^\alpha, \mathbf{v}^\alpha \in V^\alpha$, we define the forms of virtual works by

$$a^\alpha(\boldsymbol{u}^\alpha, \mathbf{v}^\alpha) = \int_{\Omega^\alpha} \sigma^\alpha(\boldsymbol{u}^\alpha)\epsilon(\mathbf{v}^\alpha)\mathrm{d}x, \qquad \ell^\alpha(\mathbf{v}^\alpha) = \int_{\Omega^\alpha} \mathbf{f}^\alpha \mathbf{v}^\alpha \,\mathrm{d}x + \int_{\Gamma_N^\alpha} \mathbf{g}^\alpha \mathbf{v}^\alpha \,\mathrm{d}\Gamma.$$

For each body, we define the total potential energy functional J^α by

$$J^\alpha(\mathbf{v}^\alpha) = \frac{1}{2} a^\alpha(\mathbf{v}^\alpha, \mathbf{v}^\alpha) - \ell^\alpha(\mathbf{v}^\alpha), \quad \forall \mathbf{v}^\alpha \in V^\alpha$$

and, for the total potential energy of the two-body system, we set $\boldsymbol{J}(\mathbf{v}) = J^1(\mathbf{v}^1) + J^2(\mathbf{v}^2)$, for all $\mathbf{v} \in \mathbf{V}$. Assuming that $\mathrm{mes}(\Gamma_D^\alpha) > 0$, the functional $\mathbf{J}$ is convex, G-differentiable and coercive on $\mathbf{V}$.

With the above preparations, the two-body contact problem (1.1)–(1.2), (1.3) can be formulated as the following constrained minimization problem

$$\mathbf{u} \in K; \quad \boldsymbol{J}(\mathbf{u}) \le \boldsymbol{J}(\mathbf{v}), \quad \forall \mathbf{v} \in K. \tag{2.1}$$

Since K is a nonempty convex and closed subset of $\mathbf{V}$, there exists a unique solution to (2.1).

Let us introduce an auxiliary unknown $\boldsymbol{q} = (q^1, q^2)$. Following Glowinski and Le Tallec [2], we introduce the set

$$C = \left\{\boldsymbol{q} = (q^1, q^2) \in (L^2(\Gamma_c))^2,\; [\boldsymbol{q}] - g \le 0 \text{ on } \Gamma_c\right\},$$

where $[\boldsymbol{q}] = q^1 - q^2$. Its characteristic functional $I_C : H = (L^2(\Gamma_c))^2 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$I_C(\boldsymbol{q}) = \begin{cases} 0 & \text{if } \boldsymbol{q} \in C \\ +\infty & \text{if } \boldsymbol{q} \notin C. \end{cases}$$

Since C is convex and nonempty ($0 \in C$), I_C is proper convex and lower semi-continuous on H. It is obvious that (2.1) is equivalent to the following constrained minimization problem

Find $(\mathbf{u}, \boldsymbol{p}) \in \mathbf{V} \times H$ such that

$$\boldsymbol{J}(\mathbf{u}) + I_C(\boldsymbol{p}) \le \boldsymbol{J}(\mathbf{v}) + I_C(\boldsymbol{q}) \quad \forall (\mathbf{v}, \boldsymbol{q}) \in \mathbf{V} \times H, \tag{2.2}$$

$$\boldsymbol{u}_n^\alpha - p^\alpha = 0 \quad \text{on } \Gamma_c, \alpha = 1, 2, \tag{2.3}$$

where $\boldsymbol{u}_n^\alpha = \boldsymbol{u}^\alpha \cdot \mathbf{n}$ with $\mathbf{n}$ the unit outward normal to Γ_c (from Ω^1 side). To (2.2)–(2.3) we associate the augmented Lagrangian functional $\mathscr{L}_r$ defined, on $\mathbf{V} \times H \times H$, by

$$\mathscr{L}_r(\mathbf{v}, \boldsymbol{q}; \boldsymbol{\mu}) = \boldsymbol{J}(\mathbf{v}) + I_C(\boldsymbol{q}) + \sum_{\alpha=1}^{2} (\mu^\alpha, \mathbf{v}_n^\alpha - q^\alpha)_{\Gamma_c} + \frac{r}{2} \sum_{\alpha=1}^{2} \|\mathbf{v}_n^\alpha - q^\alpha\|_{0,\Gamma_c}^2, \tag{2.4}$$

where $r > 0$ is the penalty parameter and $\boldsymbol{\mu} = (\mu^1, \mu^2)$. Setting $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$, the corresponding saddle-point problem is

Find $((\mathbf{u}, \boldsymbol{p}), \boldsymbol{\lambda}) \in \mathbf{V} \times H \times H$ such that

$$\mathscr{L}_r(\mathbf{u}, \boldsymbol{p}; \boldsymbol{\mu}) \le \mathscr{L}_r(\mathbf{u}, \boldsymbol{p}; \boldsymbol{\lambda}) \le \mathscr{L}_r(\mathbf{v}, \boldsymbol{q}; \boldsymbol{\lambda}), \quad \forall ((\mathbf{v}, \boldsymbol{q}), \boldsymbol{\lambda}) \in \mathbf{V} \times H \times H. \tag{2.5}$$

A saddle-point of $\mathscr{L}_r$ can be determined by a standard Uzawa method for augmented Lagrangian, see e.g. [3]. The main difficulty with the standard Uzawa method is that it leads to the coupling of subdomains (Ω^1 and Ω^2) and unknowns ($\mathbf{u}$ and $\boldsymbol{p}$). By introducing the auxiliary unknowns $\boldsymbol{p}$, we have implicitly split the problem into a "linear part" (subproblem in $\mathbf{u}$) and a "nonlinear part" (subproblem in $\boldsymbol{p}$). Furthermore, the displacements fields on subdomains Ω^1 and Ω^2 are now linked only through the auxiliary unknowns $\boldsymbol{p}$. To take advantage of these properties, a quite natural method consists of using a Uzawa block relaxation method.