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# A production–inventory model of HMMS on time scales

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**Abstract**

The aim of this work is to investigate the optimal production and inventory paths of HMMS type models (proposed by Holt, Modigliani, Muth and Simon) on complex time domains. Time scale calculus which is a rapidly growing theory is a main tool for solving and for analyzing the model. This work will enrich management and economics by providing a flexible and capable modelling technique.


*Keywords:* Calculus of variations; Time scales; Nabla derivative; Dynamic models; Optimization

## 1. Introduction

Time scale calculus is a new and exciting mathematical theory, first introduced by Stefan Hilger in his Ph.D. thesis [8], that unites two existing approaches to dynamic modelling – difference and differential equations – into a general framework called dynamic models on time scales. Since time scale calculus can be used to model dynamic processes whose time domains are more complex than the set of integers or real numbers, dynamic modelling in economy will provide a flexible and capable modelling technique for economists. Time scale calculus is very much a work in progress. For example, Atici, Biles and Lebedinsky [1] recently developed the calculus of variation theory that they used to analyze a simple economic model. This theory also has been developed for $\Delta$-derivatives by Bohner [5] and by Hilscher and Zeidan [9].

In this work, we will use the calculus of variation theory as a tool for studying the HMMS model on time scales. This model first appeared in the book by Holt, Modigliani, Muth and Simon [10] in the discrete version. They considered the problem of minimizing a cost function with constant unit costs of production and storage:

$$C = \sum_{t=1}^{\mathrm{T}} [C_P(P_t - \hat{P})^2 + C_I(I_t - \hat{I})^2] \tag{1.1}$$

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where $C_P$ is the regular time cost per unit and $C_I$ is the cost of holding a unit for one period of time. Production and finished-goods inventory must satisfy the inventory balance conditions:

$$I_t = I_{t-1} + P_t - S_t$$

where $t = 1, 2, \ldots, T$ and $P_t$, $S_t$ and $I_t$ are production rate, demand rate and inventory rate at time $t$, respectively.

In 1998, Dobos studied the continuous version of the HMMS model by taking nonnegative discount rate ($\alpha$) into account [6]:

$$C = \int_0^{\mathrm{T}} \mathrm{e}^{-\alpha t} \left[ \frac{a}{2}(I - \hat{I})^2 + \frac{b}{2}(P - \hat{P})^2 \right] \mathrm{d}t \tag{1.2}$$

subject to

$$I'(t) = P(t) - S(t) \quad \text{with } I(0) = I_0,$$

where the quantities are defined as

$I(t)$ inventory level at time $t$ (control variable);
$P(t)$ production rate at time $t$ (state variable);
$S(t)$ demand rate at time $t$; positive and continuously $\nabla$-differentiable;
$\hat{I}(t)$ inventory size goal level at time $t$;
$\hat{P}(t)$ production rate goal level at time $t$;
$\alpha$ constant, nonnegative discount rate;
$a$ constant positive, inventory holding costs coefficient;
$b$ constant positive, production costs coefficient;
$T$ length of planning horizon.

In [6], the continuous model was also studied under environmental constraints by Dobos. Later the same author investigated the optimal control problem with two state variables, inventory and store, with three control variables, rate of manufacturing, remanufacturing and disposal in [7]. [11–14] are good references for further reading about the importance of the use of the HMMS models in economy.

The work is organized as follows. In Section 2, we present some basic definitions and state some important theorems of time scale calculus to make the article self-contained. In Section 3, we introduce the HMMS model on time scales. This section provides a good application of solving a system of dynamic equations on time scales. In Section 4, we give some examples to represent the optimal path depending on the demand rate. Graphs of optimal production and inventory for various time scales follow.

## 2. Basic definitions on time scales

Let $\mathbb{T}$ be a time scale (a nonempty closed subset of $\mathbb{R}$), $[a, b]$ be the closed and bounded interval in $\mathbb{T}$, i.e., $[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$ and $a, b \in \mathbb{T}$. For the readers' convenience, we state a few basic definitions on a time scale $\mathbb{T}$.

Obviously a time scale $\mathbb{T}$ may or may not be connected. Therefore we have the concept of *forward* and *backward jump operators* as follows: Define $\sigma, \rho : \mathbb{T} \mapsto \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, then $t \in \mathbb{T}$ is called *right-dense, right-scattered, left-dense, left-scattered*, respectively. The set $\mathbb{T}_\kappa$ which is derived from $\mathbb{T}$ is as follows: If $\mathbb{T}$ has a right-scattered minimum $t_1$, then $\mathbb{T}_\kappa = \mathbb{T} - \{t_1\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. We also define the *backwards graininess function* $\nu : \mathbb{T}_\kappa \mapsto [0, \infty)$ as $\nu(t) = t - \rho(t)$.

The following two definitions and related theorems can be found in the paper by Atici and Guseinov [2].

**Definition 2.1.** If $f : \mathbb{T} \mapsto \mathbb{R}$ is a function and $t \in \mathbb{T}_\kappa$, then we define the *nabla derivative* of $f$ at a point $t$ to be the number $f^\nabla(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood of $U$ of $t$ such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon |\rho(t) - s|,$$