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A surjectivity result for quasibounded operators

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ABSTRACT

Using a degree theory for countably 1-contractive operators, we show a surjectivity theorem for such quasibounded operators. Moreover, the existence of an eigenvalue for these operators is presented.

1. Introduction

Let f be a continuous operator from a real Banach space E into itself. The problem of finding solutions for an equation of the form

$$x - f(x) = y \quad (y \in E) \quad \text{or} \quad f(x) = \lambda x \quad (\lambda \in \mathbb{R})$$

is one of the most important topics in nonlinear analysis. It was proved in [1] that if f is condensing and strictly quasibounded, then $I - f$ is surjective, where I is the identity operator on E. It is known that there is a 1-set contraction which is not condensing; see e.g. [2]. In order to obtain a surjectivity result for 1-set contractions, an additional condition might be inevitable, such as the closedness of $I - f$, as in [3].

This work deals with the above problem for a large class of operators that includes 1-set contractions, called countably 1-contractive operators. The use of countable sets was initiated by Daher [4] and it turned out in [5] that countability, e.g. a sequence of approximate solutions, could be sufficient for finding solutions of nonlinear differential equations. In view of the applications, Väth [6] established a fixed point index theory for countably condensing operators.

In Section 2, we introduce a degree theory for countably 1-contractive operators which is deduced from the index theory for countably condensing operators, as a degree theory for 1-set contractions based on Nussbaum's index theory [7] was discussed in [3]. In Section 3, we use the degree theory to prove that if f is countably 1-contractive and strictly quasibounded, then the equation $x - f(x) = y$ is solvable for every y, provided that $(I - f)(\overline{B}(0, r))$ is closed for each $r > 0$; see Theorem 1. Moreover, it is shown that under suitable conditions on f and λ the eigenvalue problem $f(x) = \lambda x$ is solvable.

In what follows, E will always be a real Banach space. Given a subset Ω of E, the closure, the boundary, and the convex hull of Ω in E are denoted by $\overline{\Omega}$, $\partial\Omega$, and $\text{co}\,\Omega$, respectively.

A functional $\gamma : \{M \subseteq E : M \text{ is bounded}\} \to [0, \infty)$ is said to be a *measure of noncompactness* on E if it satisfies the following properties:

(1) $\gamma(\overline{M}) = \gamma(M)$;
(2) $\gamma(\text{co}\,M) = \gamma(M)$;

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(3) $\gamma(M) = 0$ if and only if M is relatively compact;
(4) $\gamma(M \cup N) = \max\{\gamma(M), \gamma(N)\}$;
(5) $\gamma(M + N) \leq \gamma(M) + \gamma(N)$; and
(6) $\gamma(\alpha M) = \alpha\gamma(M)$ for every nonnegative real number α.

In this case, (4) implies that $\gamma(N) \leq \gamma(M)$ if $N \subseteq M$. Note that the Kuratowski or the Hausdorff measure of noncompactness has the above properties; see [2].

Let Ω be a subset of E and γ a measure of noncompactness on E, and $k \geq 0$. A continuous operator $f : \Omega \to E$ is said to be:

(1) *countably k-contractive* if $\gamma(f(C)) \leq k\gamma(C)$ for each countable bounded set $C \subseteq \Omega$;
(2) *countably condensing* if $\gamma(f(C)) < \gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$;
(3) *k-set-contractive* if $\gamma(f(B)) \leq k\gamma(B)$ for all bounded sets $B \subseteq \Omega$;
(4) *condensing* if $\gamma(f(B)) < \gamma(B)$ for all bounded sets $B \subseteq \Omega$ with $\gamma(B) > 0$.

A continuous homotopy $h : [0, 1] \times \Omega \to E$ is said to be:

(1) *countably 1-contractive* if $\gamma(h([0, 1] \times C)) \leq \gamma(C)$ for each countable bounded set $C \subseteq \Omega$;
(2) *countably condensing* if $\gamma(h([0, 1] \times C)) < \gamma(C)$ for each countable bounded set $C \subseteq \Omega$ with $\gamma(C) > 0$.

Note that every k-set-contractive operator is countably k-contractive and every countably condensing operator is countably 1-contractive.

2. Degree theory

We introduce a degree theory for countably 1-contractive operators which is based on the fixed point theory for countably condensing operators developed in [6], as in [3] for 1-set contractions.

Let Ω be a bounded open subset of a real Banach space $(E, \|\cdot\|)$ and $f : \overline{\Omega} \to E$ a countably 1-contractive operator such that $f(\partial\Omega)$ is bounded and

$$\|(I - f)(x)\| \geq c \quad \text{for all } x \in \partial\Omega \text{ and for some } c > 0. \tag{2.1}$$

If $g : \overline{\Omega} \to E$ is any countably k-contractive operator with $0 \leq k < 1$ such that

$$\|f(x) - g(x)\| < c \quad \text{for all } x \in \partial\Omega, \tag{2.2}$$

then $g(x) \neq x$ for all $x \in \partial\Omega$ and g is obviously countably condensing. In view of the fixed point index for countably condensing operators (see [6]), ind (g, Ω) is well defined and one uses this degree to define the degree of the operator $I - f$ on Ω over f through the relation

$$\deg(I - f, \Omega, 0) = \text{ind}\,(g, \Omega). \tag{2.3}$$

To justify the above definition, we first observe that a countably k-contractive operator $g : \overline{\Omega} \to E$ with $0 \leq k < 1$ for which (2.2) holds always exists. Indeed, it follows from

$$\|f(x)\| \leq m \quad \text{for all } x \in \partial\Omega \text{ and for some } m > 0$$

that for any $k \in (1 - cm^{-1}, 1)$ with $k \geq 0$ the operator $g := kf : \overline{\Omega} \to E$ is countably k-contractive and we have

$$\|f(x) - g(x)\| = (1 - k)\|f(x)\| \leq (1 - k)m < c \quad \text{for all } x \in \partial\Omega.$$

Next, $\deg(I - f, \Omega, 0)$ given by (2.3) is independent of the choice of g. In fact, if $g_1 : \overline{\Omega} \to E$ is another countably k_1-contractive operator with $0 \leq k_1 < 1$ for which (2.2) holds, then the continuous operator $h : [0, 1] \times \overline{\Omega} \to E$ defined by

$$h(t, x) := tg(x) + (1 - t)g_1(x) \quad \text{for } (t, x) \in [0, 1] \times \overline{\Omega}$$

is countably condensing, since for each countable set $C \subseteq \overline{\Omega}$ with $\gamma(C) > 0$ we have

$$\begin{aligned}\gamma\,(h([0, 1] \times C)) &\leq \gamma\,(\text{co}[g(C) \cup g_1(C)]) = \max\{\gamma(g(C)), \gamma(g_1(C))\} \\ &\leq \max\{k, k_1\}\gamma(C) < \gamma(C).\end{aligned}$$

Moreover, we have $h(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial\Omega$ because it follows from (2.1) and (2.2) that

$$\begin{aligned}\|x - h(t, x)\| &= \|x - f(x) + t(f(x) - g(x)) + (1 - t)(f(x) - g_1(x))\| \\ &\geq \|x - f(x)\| - t\|f(x) - g(x)\| - (1 - t)\|f(x) - g_1(x)\| \\ &> c - tc - (1 - t)c = 0.\end{aligned}$$

Hence the homotopy invariance of the index for countably condensing operators implies that

$$\text{ind}\,(g, \Omega) = \text{ind}\,(h(1, \cdot), \Omega) = \text{ind}\,(h(0, \cdot), \Omega) = \text{ind}\,(g_1, \Omega).$$

Thus, $\deg(I - f, \Omega, 0)$ is well defined.

The above degree has the following basic properties which will later be needed.