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On the solvability of two-point, second-order boundary value problems

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Abstract

We gain solvability of a system of nonlinear, second-order ordinary differential equations subject to a range of boundary conditions. The ideas involve differential inequalities and fixed point methods. In particular, maximum principles are not employed.
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1. Introduction

This work considers the existence of solutions to the nonlinear boundary value problem (BVP)

$$x'' = f(t, x, x'), \quad \text{for } t \in (0, T), \tag{1}$$
$$x \in \beta_0, \tag{2}$$

where $f : [0, T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is a continuous, nonlinear function and (2) represents any set of linear, homogeneous two-point boundary conditions, including:

$$x(0) = 0 = x(T), \quad \text{Dirichlet conditions;} \tag{3}$$
$$x'(0) = 0 = x(T), \quad \text{left focal conditions;} \tag{4}$$
$$x(0) = 0 = x'(T), \quad \text{right focal conditions;} \tag{5}$$
$$x'(0) = 0 = x'(T), \quad \text{Neumann conditions;} \tag{6}$$
$$x(0) = x(T), \qquad x'(0) = x'(T), \quad \text{periodic conditions.} \tag{7}$$

A solution to (1) and (2) is a continuously twice-differentiable function $x : [0, T] \to \mathbb{R}^n$, i.e., $x \in C^2([0, T]; \mathbb{R}^n)$, that satisfies both (1) and (2).

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The wide applications of BVPs to physics, engineering and science are well known [3, Chap.1], [1, Chap.1], and these applications naturally motivate a deeper theoretical study of the subject. We note that maximum principle techniques, including the much celebrated use of upper and lower solutions and their generalization to systems of equations, have dominated the field of solvability theory for BVPs. This work pursues an alternative approach. Instead of using maximum principles, we employ a general growth condition motivated by the classical and influential work of Hartman [8].

Using novel differential inequalities and standard fixed point methods, we establish an abstract existence result for (1) and (2) in Section 2. Sections 3 and 4 then demonstrate some applications of the aforementioned abstract result to a variety of BVPs. As for notation, if $y, z \in \mathbb{R}^n$, then $\langle y, z\rangle$ denotes their usual inner product and $\|z\|$ denotes the Euclidean norm of z. We adopt the standard norm for elements v of $C^1([0, T]; \mathbb{R}^n)$, namely

$$\|v\|_1 := \max\left\{\max_{t\in[0,T]} \|v(t)\|, \max_{t\in[0,T]} \|v'(t)\|\right\}.$$

For more on solvability to BVPs, including modern and classical approaches, see [1–11,13–18].

2. An abstract solvability result

Our main abstract existence result is

Theorem 2.1. *Let α, r, K and N be non-negative constants and let f be continuous. Suppose: that the linear BVP*

$$x'' - rx = 0 \quad \text{for } t \in (0, T), x \in \beta_0 \tag{8}$$

has only the zero solution; that

$$\|f(t, p, q) - rp\| \leq 2\alpha[\langle p, f(t, p, q)\rangle + \|q\|^2] + K \tag{9}$$

for all $(t, p, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$; and that (2) implies

$$|\langle x(T), x'(T)\rangle - \langle x(0), x'(0)\rangle| \leq N. \tag{10}$$

Then (1) and (2) has at least one solution.

Proof. Consider the following BVP, which is equivalent to (1) and (2):

$$x'' - rx = f(t, x, x') - rx, \quad t \in (0, T), \tag{11}$$

$$x \in \beta_0. \tag{12}$$

Since (8) has only the zero solution, there exists a unique, continuously differentiable Green's function $G: [0, T] \times [0, T] \to \mathbb{R}$ such that (11) and (12) may be equivalently reformulated as the integral equation

$$x(t) = \int_0^T G(t, s)\left[f(s, x(s), x'(s)) - rx(s)\right] \mathrm{d}s, \quad t \in [0, T]. \tag{13}$$

We therefore define $H : C^1([0, T]; \mathbb{R}^n) \to C^1([0, T]; \mathbb{R}^n)$ by

$$(Hx)(t) := \int_0^T G(t, s)\left[f(s, x(s), x'(s)) - rx(s)\right] \mathrm{d}s, \quad t \in [0, T] \tag{14}$$

and consider the family of equations

$$u = \lambda Hu, \quad \lambda \in (0, 1). \tag{15}$$

Since Hx actually belongs to $C^2([0, T]; \mathbb{R}^n)$ for each $x \in C^1([0, T]; \mathbb{R}^n)$, it follows from the compact embedding of $C^2([0, T]; \mathbb{R}^n)$ into $C^1([0, T]; \mathbb{R}^n)$ that H is a compact map.

We will apply the Schaefer Fixed Point Theorem [12, Theorem 4.4.10] to prove that H has at least one fixed point in $C^1([0, T]; \mathbb{R}^n)$. Since $H : C^1([0, T]; \mathbb{R}^n) \to C^1([0, T]; \mathbb{R}^n)$ is compact, it remains to verify that all solutions to (15) are bounded independently of λ. With this in mind, suppose that u satisfies (15) and define

$$G_0 := \max_{(t,s)\in[0,T]\times[0,T]} |G(t, s)|.$$

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