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The decomposition method for Cauchy reaction–diffusion problems

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Abstract

In this paper, the solution of Cauchy problems for the reaction–diffusion equation is obtained using the decomposition method. In the case when the reaction parameter is time-dependent only, an analytical solution in series form can be derived, otherwise symbolic numerical computations may need to be performed.

Keywords: Cauchy problems; Decomposition method; Reaction–diffusion equation; Reaction parameter; Series solution

1. Introduction

In this paper, we consider the one-dimensional, time-dependent reaction–diffusion equation

$$\frac{\partial w}{\partial t}(x,t) = D\frac{\partial^2 w}{\partial x^2}(x,t) + p(x,t)w(x,t), \quad (x,t) \in \Omega \subset R^2, \tag{1}$$

where w is the concentration, p is the reaction parameter and $D > 0$ is the diffusion coefficient, subject to the initial or boundary conditions

$$w(x,0) = g(x), \quad x \in R \tag{2}$$

$$w(0,t) = f_0(t), \qquad \frac{\partial w}{\partial x}(0,t) = f_1(t), \quad t \in R. \tag{3}$$

The problem given by Eqs. (1) and (2) is called the characteristic Cauchy problem in the domain $\Omega = R \times R_+$, whilst the problem given by Eqs. (1) and (3) is called the non-characteristic Cauchy problem in the domain $\Omega = R_+ \times R$.

The solution of these problems is attempted using the Adomian decomposition method (ADM), as described next.

2. The decomposition method

Defining the partial differential operators $L_t = \partial/\partial t$ and $L_{xx} = \partial^2/\partial x^2$, then Eq. (1) can be rewritten as

$$L_t w = DL_{xx}w + pIw, \tag{4}$$

where I is the identity operator.

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Let us formally define the left-inverse integral operators, [1,2],

$$L_t^{-1} = \int_0^t \mathrm{d}t', \qquad L_{xx}^{-1} = \int_0^x \mathrm{d}x' \int_0^{x'} \mathrm{d}x''. \tag{5}$$

We then seek the solution of Eq. (4) in the form of the decomposition series

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t), \quad (x,t) \in \Omega, \tag{6}$$

where the components $(w_n)_{n\geq 0}$ satisfy the recursive relationships

$$w_0(x,t) = g(x), \qquad w_{n+1}(x,t) = L_t^{-1}[DL_{xx} + p(x,t)I]w_n(x,t), \quad n \geq 0 \tag{7}$$

for the characteristic Cauchy problem (1) and (2), and

$$w_0(x,t) = f_0(t) + xf_1(t), \qquad w_{n+1}(x,t) = \frac{1}{D}L_{xx}^{-1}[L_t - p(x,t)I]w_n(x,t), \quad n \geq 0 \tag{8}$$

for the non-characteristic Cauchy problem (1) and (3).

2.1. *The case p = constant*

In this case, Eq. (1) becomes

$$\frac{\partial w}{\partial t}(x,t) = D\frac{\partial^2 w}{\partial x^2}(x,t) + pw(x,t), \quad (x,t) \in \Omega. \tag{9}$$

Applying (7) we obtain

$$w_1(x,t) = L_t^{-1}[DL_{xx} + pI]w_0(x,t) = (Dg''(x) + pg(x))t,$$
$$w_2(x,t) = L_t^{-1}[DL_{xx} + pI]w_1(x,t) = (D^2 g''''(x) + 2Dpg''(x) + p^2 g(x))\frac{t^2}{2!},$$

and, in general, we observe that

$$w_n(x,t) = \left(\sum_{l=0}^{n} C_n^l D^l p^{n-l} g^{(2l)}(x)\right)\frac{t^n}{n!}, \quad n \geq 0,$$

where $C_n^l = \frac{n!}{l!(n-l)!}$. Then based on (6) we obtain the ADM partial t-solution of the problem (2) and (9) given by

$$w(x,t) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} C_n^l D^l p^{n-l} g^{(2l)}(x)\right)\frac{t^n}{n!}, \quad (x,t) \in R \times R_+. \tag{10}$$

Applying now (8) we obtain

$$w_1(x,t) = \frac{1}{D}L_{xx}^{-1}[L_t - pI]w_0(x,t) = \frac{1}{D}\left[(f_0'(t) - pf_0(t))\frac{x^2}{2!} + (f_1'(t) - pf_1(t))\frac{x^3}{3!}\right],$$
$$w_2(x,t) = \frac{1}{D}L_{xx}^{-1}[L_t - pI]w_1(x,t) = \frac{1}{D^2}\left[(f_0''(t) - 2pf_0'(t) + p^2 f_0(t))\frac{x^4}{4!} + (f_1''(t) - 2pf_1'(t) + p^2 f_1(t))\frac{x^5}{5!}\right],$$

and, in general, we observe that

$$w_n(x,t) = \frac{1}{D^n}\left(\sum_{l=0}^{n} C_n^l (-p)^{n-l} f_0^{(l)}(t)\frac{x^{2n}}{(2n)!} + \sum_{l=0}^{n} C_n^l (-p)^{n-l} f_1^{(l)}(t)\frac{x^{2n+1}}{(2n+1)!}\right), \quad n \geq 0.$$