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Non-existence of positive solutions of some elliptic equations in positive-type domains

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Abstract

In this work we prove a non-existence result for a positive solution of a Dirichlet boundary value problem in a class of domains that are more general than star-shaped ones. Our result extends some earlier non-existence results for the star-shaped case. We also present an example of a domain which satisfies our conditions but not star-shaped essentially.

Keywords: Non-existence; Positive solution; Pohozaev-type identity; Star-shaped domain; Positive-type domain

1. A non-existence result for elliptic equations

Let $\Omega \subset \mathbf{R}^n$ be a smooth domain; consider the following elliptic boundary value problem:

$$\begin{cases} -\Delta u = u|u|^{\alpha-1} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

If Ω is star-shaped, based on the work of Pohozaev [5], there have been many non-existence results for positive solution of (1.1) with supercritical growth (see [5–7]). Recently some non-existence results appeared for the case that Ω is not necessarily star-shaped; see for example [4]. It is still an interesting problem to consider the non-existence results related to the growth and domain geometry. In [1–3], the author of this work introduced a class of hypersurfaces that are more general than star-shaped ones. Motivated by the ideas of [1–3], in this work we discuss the non-existence of a positive solution of (1.1) in a new class of domains called positive-type domains. Our results extend some earlier related works for the star-shaped case.

We firstly consider the solutions of the elliptic boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $f \in C(R^1, R^1)$, $\Omega \subset \mathbf{R}^n$ is a domain with smooth boundary $\partial\Omega$. Let $F(x) = \int_0^x f(s)\mathrm{d}s$. Denote by $\langle\cdot,\cdot\rangle$ the usual inner product of $\mathbf{R}^n$ and $\nu(x)$ the outward normal to $\partial\Omega$ at x.

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Lemma 1.1. *Suppose that $V(x) = (V_1(x), \ldots, V_n(x))$ is a C^1 vector field on $\mathbf{R}^n$ and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of (1.2). Then*

$$\int_\Omega u \operatorname{div} V(x) \mathrm{d}x = -\int_\Omega \langle V(x), \nabla u\rangle \mathrm{d}x,$$
$$\int_\Omega F(u) \operatorname{div} V(x) \mathrm{d}x = -\int_\Omega f(u) \langle V(x), \nabla u\rangle \mathrm{d}x.$$

Proof. Since $u(x) = 0$ and $F(u(x)) = 0$ for $x \in \partial\Omega$, by the divergence theorem, one has

$$\begin{aligned} 0 &= \int_{\partial\Omega} \langle u(x)V(x), \nu(x)\rangle \mathrm{d}s = \int_\Omega \operatorname{div}(u(x)V(x))\mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}(u(x)V_i(x)) = \int_\Omega u \operatorname{div} V(x)\mathrm{d}x + \int_\Omega \langle V(x), \nabla u\rangle \mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{\partial\Omega} \langle F(u(x))V(x), \nu(x)\rangle \mathrm{d}s = \int_\Omega \operatorname{div}(F(u(x))V(x))\mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}(F(u(x))V_i(x)) = \int_\Omega F(u) \operatorname{div} V(x)\mathrm{d}x + \int_\Omega f(u)\langle V(x), \nabla u\rangle \mathrm{d}x. \end{aligned}$$

The proof is complete. □

On the basis of Lemma 1.1, we prove an identity for the solutions of (1.2).

Theorem 1.2. *Suppose $V(x)$ is a linear vector field on $\mathbf{R}^n$ with the form*

$$V(x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} x.$$

If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution of (1.2), then

$$2\sum_{i=1}^n a_{ii} \int_\Omega F(u)\mathrm{d}x - \sum_{i=1}^n a_{ii} \int_\Omega u f(u)\mathrm{d}x + 2\int_\Omega \langle V(\nabla u), \nabla u\rangle \mathrm{d}x = \int_{\partial\Omega} \langle V(x), \nu(x)\rangle |\nabla u|^2 \mathrm{d}s. \tag{1.3}$$

Proof. By divergence theorem and Lemma 1.1,

$$\begin{aligned} &\int_{\partial\Omega} \langle V(x), \nu(x)\rangle |\nabla u|^2 \mathrm{d}s = \int_{\partial\Omega} \langle |\nabla u|^2 V(x), \nu(x)\rangle \mathrm{d}s = \int_\Omega \operatorname{div}(|\nabla u|^2 V(x))\mathrm{d}x \\ &= \int_\Omega \left(\sum_{i=1}^n a_{ii} |\nabla u|^2 + 2\sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \sum_{i=1}^n V_i(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \right) \mathrm{d}x \\ &= \sum_{i=1}^n a_{ii} \int_\Omega u f(u)\mathrm{d}x - 2\int_\Omega \sum_{j=1}^n u \frac{\partial}{\partial x_j} \left(\sum_{i=1}^n V_i(x) \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \mathrm{d}x \\ &= \sum_{i=1}^n a_{ii} \int_\Omega u f(u)\mathrm{d}x - 2\int_\Omega u \sum_{j=1}^n \sum_{i=1}^n \left(a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + V_i(x) \frac{\partial}{\partial x_i} \frac{\partial^2 u}{\partial x_j^2} \right) \mathrm{d}x \\ &= \sum_{i=1}^n a_{ii} \int_\Omega u f(u)\mathrm{d}x - 2\int_\Omega u \operatorname{div}(V(\nabla u))\mathrm{d}x - 2\int_\Omega \sum_{j=1}^n \sum_{i=1}^n u V_i(x) \frac{\partial}{\partial x_i} \frac{\partial^2 u}{\partial x_j^2} \mathrm{d}x \\ &= \sum_{i=1}^n a_{ii} \int_\Omega u f(u)\mathrm{d}x + 2\int_\Omega \langle V(\nabla u), \nabla u\rangle \mathrm{d}x + 2\int_\Omega \sum_{j=1}^n \sum_{i=1}^n \frac{\partial}{\partial x_i}(u V_i(x)) \frac{\partial^2 u}{\partial x_j^2} \mathrm{d}x \end{aligned}$$