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A delayed marine bacteriophage infection model☆

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Abstract

A marine bacteriophage infections model with stage structure is studied. Necessary and sufficient conditions for the extinction and permanence of the system are obtained, which enrich and improve the corresponding results given by S.A. Gourley and Y. Kuang [A delay reaction–diffusion model of the spread of bacteriophage infection, SIAM J. Appl. Math. 65 (2005) 550–566].

Keywords: Marine bacteriophage infection; Delay; Stage structure; Necessary and sufficient conditions; Permanence

1. The model

Dynamics of marine bacteriophage infections have received much attention and many excellent results have been obtained: In [3], Beretta and Kuang proposed a mathematical model for the marine bacteriophage infection and analyzed its essential mathematical features. Later, they [4] extended the model of [3] in modeling of the latent period by suitable delay terms modeling. Recently, Gourley and Kuang [7] considered the following marine bacteriophage infection model with stage structure:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \alpha S(t)\left(1 - \dfrac{S(t)}{\gamma}\right) - KS(t)P(t), \\ \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = -\mu_p P(t) - mP^2(t) - KS(t)P(t) + bKS(t-T)P(t-T)\mathrm{e}^{-\mu_i T}, \end{cases} \tag{1.1}$$

where $S(t)$ is the density (i.e. number of bacteria per liter) of susceptible bacteria, $P(t)$ is the density (number of viruses per liter) of viruses (phages). T is a constant and denotes the latency time. The initial conditions for (1.1) are

$$\begin{cases} S(s) = S^0(s) \geq 0, \quad s \in [-T, 0], \text{ with } S^0(0) > 0, \\ P(s) = P^0(s) \geq 0, \quad s \in [-T, 0], \text{ with } P^0(0) > 0, \end{cases} \tag{1.2}$$

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where S^0 and P^0 are prescribed continuous functions. In [7], Gourley and Kuang studied system (1.1) with (1.2) and the existence with a reaction–diffusion version of (1.1). For system (1.1) with (1.2), they obtained the following sharp result on the extinction of viruses $P(t)$:

Theorem 1 (*Gourley and Kuang, [7], Theorem 1*). *Assume*

$$b\mathrm{e}^{-\mu_i T} \leq 1 + \frac{\mu_p}{\gamma K}. \tag{1.3}$$

Then any solution of (1.1) with (1.2) satisfies $\lim_{t\to\infty}(S(t), P(t)) = (\gamma, 0)$.

However, there is one question was left unsolved: Under which conditions will the susceptible bacteria $S(t)$ and viruses $P(t)$ coexist permanently?

In this work, we will study the permanence and extinction of system (1.1). To get the permanence, we engage Hale and Waltmann's persistence theory ([8]; for it applications, we refer the reader to Thieme [17,18] and Magal and Zhao [16], Liu et al. [10,19,15]). We also use similar techniques for global stability of stage-structured models (see Aiello et al. [1], Al-Omari et al. [2], Gourley and Kuang [6], Liu et al. [10–14,19,15], Beretta et al. [5] and Kuang [9]). We improve Theorem 1 by obtaining the necessary and sufficient conditions for extinction and permanence of system (1.1). Our main results are as follows:

Theorem 2. *System (1.1) with (1.2) is permanent if and only if it satisfies*

$$b\mathrm{e}^{-\mu_i T} > 1 + \frac{\mu_p}{\gamma K}. \tag{1.4}$$

Theorem 3. *The axis equilibrium* $(\gamma, 0)$ *of system (1.1) with (1.2) is globally attractive if and only if (1.3) holds.*

Remark 1. Theorem 3 suggests that (1.3) is also the necessary condition for extinction of viruses $P(t)$; this directly improves Theorem 1 in Gourley and Kuang [7].

Remark 2. By (3.2) in [7], the positive equilibrium of system (1.1) exists if and only if (1.4) holds true. Thus by Theorem 2, the permanence of (1.1) is equivalent to the existence of its positive equilibrium. The biological meanings from conditions (1.4) and (1.3) are: if and only if the difference of the recruitment rate of the virus and its leaving rate (by infecting bacteria) at the peak of bacterial abundance γ is larger than its death rate μ_p, then the viruses coexist with susceptible bacteria permanently; if the contrary, then the viruses face extinction.

2. Proof of main results

To prove the above main results, we need some preliminary results. We have

Lemma 1 (*Gourley and Kuang, [7], Proposition 1*). *Solutions of (1.1) with (1.2) satisfy* $S(t) > 0,\ P(t) > 0$ *for all* $t > 0$.

Lemma 2 (*Liu et al. [14], Lemma 2*). *For equation*

$$\dot{x}(t) = bx(t-\tau) - a_1x(t) - a_2x^2(t), \quad (a_1 \geq 0, a_2, b, \tau > 0 \text{ and } x(t) > 0 \text{ for all } -\tau \leq t \leq 0),$$

we have

(i) *If* $b > a_1$, *then* $\lim_{t\to+\infty} x(t) = \frac{b-a_1}{a_2}$.
(ii) *If* $b < a_1$, *then* $\lim_{t\to+\infty} x(t) = 0$.

To prove Theorem 2, we present the persistence theory of Hale and Waltmann [8] for infinite dimensional systems as follows.

Consider a metric space X with metric d. T is a continuous semiflow on X_1, i.e., a continuous mapping $T : [0,\infty) \times X \to X$ with the following properties:

$$T_t \circ T_s = T_{t+s}, \quad t, s \geq 0; \qquad T_0(x) = x, \quad x \in X.$$