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Existence of extremal solutions for discontinuous functional integral equations

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Abstract

In this work the existence of extremal solutions of nonlinear discontinuous functional integral equations of mixed type $x(t) = q(t) + \sum_{i=1}^{3} \int_0^{\sigma_i(t)} k_i(t,s) f_i(s, x(\eta_i(s)))\,ds$ is proved under mixed Lipschitz, Carathéodory and monotonicity conditions.

Keywords: Functional integral equation; Extremal solutions

1. Functional integral equations

Let $\mathbf{R}$ be the real line. Given a closed and bounded interval $J = [0, 1]$ in $\mathbf{R}$, consider the nonlinear functional integral equation (for short FIE) of mixed type

$$x(t) = q(t) + \sum_{i=1}^{3} \int_0^{\sigma_i(t)} k_i(t,s) f_i(s, x(\eta_i(s)))\,ds, \qquad t \in J \tag{1.1}$$

where $q : J \to \mathbf{R}$, $k_i : J \times J \to \mathbf{R}$, $f_i : J \times \mathbf{R} \to \mathbf{R}$ and $\sigma_i, \eta_i : J \to J$ for $i = 1, 2, 3$.

By a solution of the FIE (1.1) we mean a function $x \in BM(J, \mathbf{R})$ that satisfies (1.1) on J, where $BM(J, \mathbf{R})$ is the space of all bounded and measurable real-valued functions on J.

The FIE (1.1) is new to the theory of nonlinear integral equations and includes several known integral equations studied earlier as special cases. The special case in the form

$$x(t) = q(t) + \int_0^{t} k_1(t,s) f_1(s, x(s))\,ds + \int_0^{1} k_2(t,s) f_2(s, x(s))\,ds, \qquad t \in J \tag{1.2}$$

has been studied extensively under mixed Lipschitz and Carathéodory conditions of nonlinearities involved in the equations. Very recently, the FIE in the form

$$x(t) = q(t) + \sum_{i=1}^{2} \int_0^{\sigma_i(t)} k_i(t,s) f_i(s, x(\eta_i(s)))\,ds, \qquad t \in J \tag{1.3}$$

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has been discussed by Dhage and Ntouyas [5] for existence of the solution under mixed Lipschitz and Carathéodory conditions and using a nonlinear alternative of Burton and Kirk [1]. To the best of our knowledge, the FIE (1.1) involving discontinuous nonlinearity has not been studied so far for the existence results. In this work, we will prove the existence of extremal solutions for discontinuous FIE (1.1) under mixed Lipschitz, Carathéodory and monotonicity conditions.

2. Auxiliary results

Let X be a normed linear space with norm $\|\cdot\|$ and let $T : X \to X$. T is called Lipschitz if there exists a constant $k > 0$ such that $\|Tx - Ty\| \le k\|x - y\|$ for all $x, y \in X$. The constant k is called a Lipschitz constant of T on X. Further if $k < 1$, then T is called a contraction on X with contraction constant k.

Again T is called **totally compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. T is called a **compact** if $\overline{T(S)}$ is a compact for a bounded subset S of X. T is called **totally bounded** if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. Finally T is called **completely continuous** if it is compact and continuous on X. Note that every compact operator is totally bounded but the converse may not be true. However, these two notions are equivalent on bounded subsets of a complete normed linear space, i.e., a Banach space X.

A non-empty closed subset K of X is called a cone if (i) $K + K \subset K$, (ii) $\lambda K \subset K$ for all $\lambda \in \mathbf{R}^+$ and (iii) $-\{K\} \cap K = \{0\}$, where 0 is a zero element of X. We define an order relation $\le$ in X with the help of the cone K as follows. Let $x, y \in X$. Then

$$x \le y \iff y - x \in K. \tag{2.1}$$

Now the normed linear space X together with the order relation $\le$ becomes an ordered normed linear space and it is denoted by (X, K). A cone K in X is called normal if the norm $\|\cdot\|$ is semi-monotone on it, i.e., if $x, y \in K$, then $\|x\| \le N\|y\|$ for some real number $N > 0$. It is known that if a cone K in X is normal, then every order bounded set in X is norm-bounded. The details of cones and their properties may be found in Guo and Lakshmikantham [6].

Let $a, b \in X$ be such that $a \le b$. Then by an order interval $[a, b]$ we mean a set in X defined by

$$[a, b] = \{x \in X \mid a \le x \le b\}. \tag{2.2}$$

Definition 2.1. A mapping $T : X \to X$ is called monotone increasing if $Tx \le Ty$, whenever $x \le y$, for all $x, y \in X$.

We use the following slight improvement of a hybrid fixed point of Dhage [2] in the sequel.

Theorem 2.1. *Let X be a Banach space and let $A, B, C : X \to X$ be three monotone increasing operators such that*

(a) *A is a contraction with contraction constant $\alpha < 1$,*
(b) *B is completely continuous,*
(c) *C is totally bounded, and*
(d) *there exist elements a and b in X such that $a \le Aa + Ba + Ca$ and $b \ge Ab + Bb + Cb$ with $a \le b$.*

Further if the cone K in X is normal, then the operator equation $Ax + Bx + Cx = x$ has a least and a greatest solution in $[a, b]$.

3. Existence of extremal solutions

We shall see the solution of the FIE (1.1) in the space $C(J, \mathbf{R})$ of continuous real-valued functions on J. Define a norm $\|\cdot\|$ in $C(J, \mathbf{R})$ by $\|x\| = \sup_{t \in J} |x(t)|$. Then $C(J, \mathbf{R})$ is a Banach space with respect to this supremum norm. Again introduce an order relation $\le$ in $C(J, \mathbf{R})$ with the help of the cone K in X defined by

$$K = \{x \in C(J, \mathbf{R}) \mid x(t) \ge 0 \,\forall\, t \in J\}. \tag{3.1}$$

Clearly K is a normal cone in X.

We need the following definitions in the rest of the work.

Definition 3.1. A function $\beta : J \times \mathbf{R} \to \mathbf{R}$ is said to be generalized Lipschitz if there exists a function $\ell \in L^1(J, \mathbf{R})$ such that $|\beta(t, x) - \beta(t, y)| \le \ell(t)|x - y|$ a.e. $t \in J$ for all $x, y \in \mathbf{R}$. The function ℓ is called the Lipschitz function of β.