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Numerical solutions of Burgers' equation with random initial conditions using the Wiener chaos expansion and the Lax–Wendroff scheme

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Abstract

The work is concerned with efficient computation of statistical moments of solutions to Burgers' equation with random initial conditions. When the Lax–Wendroff scheme is expanded using the Wiener chaos expansion (WCE), it introduces an infinite system of deterministic equations with respect to non-random Hermite–Fourier coefficients. One of important properties of the system is that all the statistical moments of the solution can be computed using simple formulae that involve only the solution of the system. The stability, accuracy, and efficiency of the WCE approach to computing statistical moments have been numerically tested and compared to those for the Monte Carlo (MC) method. Strong evidence has been given that the WCE approach is as accurate as but substantially faster than the MC method, at least for certain classes of initial conditions.

Keywords: Burgers' equation; Lax–Wendroff scheme; Wiener chaos expansion; Monte Carlo method

1. Introduction

Uncertainty is observed in many and various phenomena in engineering, physics, biology, and finance. For example, small scale effects in multi-phase flow [1,2] may not be completely known, but subject to some random environmental effects. Then, the governing equation including such uncertainty is expressed in the form of a stochastic partial differential equation (SPDE). Since its solution is random, it is important to know the statistical properties of the solution.

Wiener chaos expansion (WCE) is a Fourier expansion with respect to the randomness [3–6]. In this work, it is expanded with respect to the Gaussian random variable. The WCE of a solution to Burgers' equation, written by $v(x, t, \xi)$, is given by

$$v(x, t, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x, t)\xi_n(\xi), \tag{1}$$

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where ξ_n is the nth-order Wick polynomial. The series in the right hand side of (1) converges in $L_2(R, \mu)$, where the Gaussian measure $\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}\mathrm{d}x$. As every Fourier expansion will do, WCE separates variables. More specifically, it separates non-random variables (x, t) from the random one ξ. Then non-random Hermite–Fourier coefficients $\{v^n\}$ satisfy an infinite system of deterministic equations with nonlinearity similar to that in Burgers' equation. This system for the Hermite–Fourier coefficients is usually referred to as a *propagator*, because it governs the propagation of randomness by the deterministic dynamics of the equation. Since all the statistical moments of the solution to Burgers' equation depend on only the solution of the propagator, WCE presents an alternative to computing moments by the Monte Carlo method.

This polynomial chaos expansion has been used by many authors. Chorin [7] improves the efficiency of Monte Carlo calculations by using truncated orthogonal polynomial expansions. The spectral formulation of the deterministic finite element method has been extended to the space of random functions in [8]. Jardak et al. [9] present a new algorithm based on WCE for solving the advection equation with stochastic transport velocity. Hou et al. [10] develop numerical methods for randomly forced equations in fluid dynamics using WCE.

2. Numerical algorithm

The inner product of two real-valued functions $F(x)$ and $G(x)$ is defined by $\langle F, G\rangle_w \equiv \int_{-\infty}^{\infty} F(x)G(x)w(x)\mathrm{d}x$, where $w(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$. H_n and ξ_n, defined by $H_0(x) = 1$, $H_n(x) = (-1)^n \mathrm{e}^{x^2/2}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2/2}\right)$ and $\xi_n(x) = \frac{1}{\sqrt{n!}}H_n(x)$, are called the *Hermite* and *Wick polynomials of order* n, respectively. [3,4] show that $\{\xi_n(x)\}$ is a complete orthonormal set in $L^2(-\infty, \infty)$ with respect to $w(x)$.

Let us consider the following Burgers' equation with a random initial condition:

$$v_t + f(v)_x = 0, \qquad v(x, 0, \xi) = g(x, \xi) \tag{2}$$

where $f(v) = \frac{1}{2}v^2$ and $g(x, \xi)$ is a function of x and ξ. ξ is a Gaussian random variable $N(0, 1)$ with zero mean and unit variance. The WCE represents the solution v as $v(x, t, \xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} v^n(x, t)\xi_n(\xi)$ in (1), where $v^n(x, t) \equiv \sqrt{n!}\langle v(x, t, \xi), \xi_n(\xi)\rangle_w$ is called the *Hermite–Fourier* coefficient of order n.

The Lax–Wendroff scheme for the deterministic conservation laws, $v_t + f(v)_x = 0$, computes $v_{j,i} \equiv v(x_j, t_i)$ using

$$v_{j,i+1} = v_{j,i} - \frac{\mu}{2}(\Delta_{+x}F_{j,i} + \Delta_{-x}F_{j,i}) + \frac{\mu^2}{2}(A_{j+\frac{1}{2},i}\Delta_{+x}F_{j,i} - A_{j-\frac{1}{2},i}\Delta_{-x}F_{j,i}) \tag{3}$$

where $\mu = \frac{\Delta t}{\Delta x}$, $F_{j,i} \equiv f(v_{j,i})$ and $A_{j,i} \equiv \frac{\partial f}{\partial v}(v_{j,i})$. $\Delta_{+x}F_{j,i}$ and $\Delta_{-x}F_{j,i}$ are defined by $(F_{j+1,i} - F_{j,i})$ and $(F_{j,i} - F_{j-1,i})$, respectively. $A_{j\pm\frac{1}{2},i}$ stand for $\frac{1}{2}(A_{j,i} + A_{j\pm1,i})$. From the orthogonality, $\{H_n\}$ satisfies

$$H_\alpha(x)H_\beta(x) = \sum_{\gamma=0}^{\alpha\wedge\beta} \gamma! \binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma}(x).$$

Thus, when $v_{j,i}$ in (3) is expanded using WCE, the propagator system for each Hermite–Fourier coefficient $v^n_{j,i} \equiv v^n(x_j, t_i)$ can be solved by the following modified version of the Lax–Wendroff scheme:

$$\begin{aligned} v^n_{j,i+1} = {} & v^n_{j,i} - \frac{\mu}{4}(w^n_{j+1,i} - w^n_{j-1,i}) \\ & + \frac{\mu^2}{8}\sum_{\alpha=0}^{\infty}\{(z^{n,\alpha}_{j,i} + z^{n,\alpha}_{j+1,i})(w^\alpha_{j+1,i} - w^\alpha_{j,i}) - (z^{n,\alpha}_{j,i} + z^{n,\alpha}_{j-1,i})(w^\alpha_{j,i} - w^\alpha_{j-1,i})\}, \end{aligned} \tag{4}$$

where $w^\alpha_{j,i} = \sum_{m=0}^{\infty} \frac{1}{m!}\left(2\sum_{0\le k<\frac{\alpha}{2}} \binom{\alpha}{k} v^{m+k}_{j,i} v^{m+\alpha-k}_{j,i} + \chi_{\{\frac{\alpha}{2}=[\frac{\alpha}{2}]\}}\binom{\alpha}{\frac{\alpha}{2}}\left(v^{m+\frac{\alpha}{2}}_{j,i}\right)^2\right)$ and $z^{n,\alpha}_{j,i} = \sum_{q=\max\{0,n-\alpha\}}^{n} \frac{1}{(\alpha-n+q)!}\binom{n}{q} v^{\alpha-n+2q}_{j,i}$. $\chi_{\{a=b\}} = 1$ if $a = b$, and 0 otherwise, and $[x]$ is the smallest integer which is not smaller than x. It is important to notice that (4) separates $\{z^{n,\alpha}_{j,i}\}$ and $\{w^\alpha_{j,i}\}$, which removes unnecessary nested loops and