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Positive solutions of nonlinear second-order periodic boundary value problems

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Abstract

In this paper, we consider the nonlinear second-order periodic boundary value problem

$$u''(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, 2\pi]; \qquad u(0) = u(2\pi), \qquad u'(0) = u'(2\pi),$$

where the nonlinear term f is a Caratheodory function. By introducing two height functions concerned with f and considering the integrals of height functions on some bounded sets, we prove the existence and multiplicity of positive solutions for the problem.

Keywords: Nonlinear ordinary differential equation; Periodic boundary value problem; Positive solution; Existence; Multiplicity

1. Introduction

In this paper, we study the existence of single and multiple positive solutions of the following nonlinear second-order periodic boundary value problem

$$\begin{cases} u''(t) = f(t, u(t)), & \text{a.e. } t \in [0, 2\pi], \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \tag{P}$$

Here, a positive solution of the problem (P) means a solution u^* of (P) satisfying $u^*(t) > 0$, a.e. $t \in [0, 2\pi]$.

Throughout this paper, we assume that k is a constant satisfying $0 < k < \frac{1}{4}$ and $f : [0, 2\pi] \times (0, +\infty) \to (-\infty, +\infty)$ is a Carathéodory function, that is, f satisfies the following conditions:

(a1) For a.e. $t \in [0, 2\pi]$, $f(t, \cdot) : (0, +\infty) \to (-\infty, +\infty)$ is continuous.

(a2) For all $u \in (0, +\infty)$, $f(\cdot, u) : [0, 2\pi] \to (-\infty, +\infty)$ is measurable.

(a3) For any $0 < c < d$, there exists a nonnegative function $h_{(c,d)} \in L[0, 2\pi]$ such that $|f(t, u)| \le h_{(c,d)}(t)$, a.e. $t \in [0, 2\pi]$ and $\forall u \in [c, d]$.

In a real problem, we may choose k by the growth feature of nonlinear term f. Moreover, the conditions (a1)–(a3) imply that the problem (P) may be singular or nonsingular.

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Because of wide interests in physics and engineering, second-order periodic boundary value problems have been investigated by many authors (see [1–6]). In most real problems, only the positive solution is significant. But it is more difficult to find references for the positive solutions of (*P*). One of the reasons is that the problem (*P*) has no Green function. Recently, Torres [5] surmounted this difficult point and proved the following existence theorem.

Theorem 1.1. *Let there exist two positive numbers a, b such that*
(1) $f(t,u)+ku \geq 0$, a.e. $t \in [0, 2\pi]$, $\forall u \in [\sigma \min\{a,b\}, \max\{a,b\}]$.
(2) $f(t,u)+ku \leq \frac{1}{2\pi M}u$, a.e. $t \in [0, 2\pi]$, $\forall u \in [\sigma a, a]$.
(3) $f(t,u)+ku \geq \frac{1}{2\pi \sigma m}u$, a.e. $t \in [0, 2\pi]$, $\forall u \in [\sigma b, b]$.
Then problem (P) *has one positive solution.*

Here, $m = \frac{\cos\sqrt{k}\pi}{2\sqrt{k}\sin\sqrt{k}\pi}$, $M = \frac{1}{2\sqrt{k}\sin\sqrt{k}\pi}$, $\sigma = \cos\sqrt{k}\pi < 1$.

Theorem 1.1 is an effective tool for the positive solution of problem (*P*) if $f(t,u)/u$ are essential bounded in t on bounded closed sets $[0, 2\pi] \times [\sigma a, a]$ and $[0, 2\pi] \times [\sigma b, b]$. But Theorem 1.1 is powerless if $f(t,u)/u$ is not essential bounded in t on one of these sets.

The aim of this paper is to cancel this deficiency and improve Theorem 1.1. The idea of this work comes from Torres [5], Jiang [6] and our papers [7–12]. We will draw into two height functions to describe the growth feature of nonlinear term f. And then, we will apply Guo–Krasnosel'skii fixed point theorem of cone expansion–compression type to establish a basic existence theorem, that is Theorem 3.1. The existence theorem shows that the problem (*P*) has at least one positive solution provided the integrals of height functions are appropriate on some bounded sets. In Section 3, we will also consider the existence of multiple positive solutions. In Section 5, we will investigate the case concerning the limits of growth rates $f(t,u)/u$ at 0 or $+\infty$. Finally, we will illustrate that our improvement is true in Section 6 and explain the possibility for some extensions of this work in Section 7.

2. Preliminaries – I

Consider the Banach space $C[0, 2\pi]$ with norm $\|u\| = \max_{0 \leq t \leq 2\pi} |u(t)|$ and let

$$C^+[0,2\pi] = \{u \in C[0,2\pi] : u(t) \geq 0, 0 \leq t \leq 2\pi\},$$
$$K = \{u \in C^+[0,2\pi] : u(t) \geq \sigma\|u\|, 0 \leq t \leq 2\pi\}.$$

It is easy to check that K is a cone of nonnegative functions in $C[0, 2\pi]$. Let $\Omega(c) = \{u \in K : \|u\| < c\}$, $\partial\Omega(c) = \{u \in K : \|u\| = c\}$.

Denote

$$G(t,s) = \begin{cases} \dfrac{\cos\sqrt{k}(\pi - t + s)}{2\sqrt{k}\sin\sqrt{k}\pi}, & 0 \leq s \leq t \leq 2\pi, \\ \dfrac{\cos\sqrt{k}(\pi + t - s)}{2\sqrt{k}\sin\sqrt{k}\pi}, & 0 \leq t \leq s \leq 2\pi. \end{cases}$$

Obviously, $G(t,s) > 0$, $0 \leq t, s \leq 2\pi$. After computations, we obtain

$$m = \min_{0 \leq t,s \leq 2\pi} G(t,s), \qquad M = \max_{0 \leq t,s \leq 2\pi} G(t,s), \qquad \sigma = mM^{-1}.$$

Define the operator T as follows

$$(Tu)(t) = \int_0^{2\pi} G(t,s)[f(s,u(s)) + ku(s)]\mathrm{d}s, \quad 0 \leq t \leq 2\pi.$$

Lemma 2.1. (1) *For any $0 < r_1 < r_2$, $T : \overline{\Omega(r_2)} \setminus \Omega(r_1) \to C[0, 2\pi]$ is a compact operator.*
(2) *If $u \in K$ such that $f(t,u(t)) + ku(t) \geq 0$, a.e. $t \in [0, 2\pi]$, then $Tu \in K$.*
(3) *If $u^* \in K$ is a fixed point of T and $u^* \neq 0$, then u^* is a positive solution of* (P).