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Applied Mathematics Letters 19 (2006) 351–355
Applied Mathematics Letters
www.elsevier.com/locate/aml

Localized sampling in the presence of noise

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Received 18 April 2005; accepted 12 May 2005

Abstract

When the sampled values are corrupted by noise, error estimates for the localized sampling series for approximating a band-limited function are obtained. The result provides error bounds for practical cases including error caused by average sampling, jitter error and amplitude error.

MSC: primary 41A80, 41A30; secondary 65D25, 65G99, 94A24

Keywords: Localized sampling; Noise; Average sampling; Jitter error; Amplitude error

1. Introduction

Let $L^p(\mathbb{I})$, where $1 \le p \le \infty$ and $\mathbb{I} \subseteq \mathbb{R}$, be the space of complex-valued Lebesgue measurable functions f on $\mathbb{I}$ for which the norm, defined by $\|f\|_{p,\mathbb{I}} := \left\{\int_{\mathbb{I}} |f(x)|^p \mathrm{d}x\right\}^{1/p}$ for $1 \le p < \infty$ and $\|f\|_{\infty,\mathbb{I}} := \text{ess sup}\{|f(x)| : x \in \mathbb{I}\}$, is finite. When $\mathbb{I} = \mathbb{R}$ we drop the subscript $\mathbb{I}$ on the norm for notational simplicity. Every function $f \in L^2(\mathbb{R})$ has a Fourier transform in $L^2(\mathbb{R})$ which is defined by $\widehat{f}(\omega) := \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}x\omega}\mathrm{d}x$ for $\omega \in \mathbb{R}$. A function f in $L^2(\mathbb{R})$ is said to be band-limited, with bandwidth σ, provided that we have for $|\omega| > \sigma$ that $\widehat{f}(\omega) = 0$. We denote the totality of such functions by $B_\sigma := \{f \in L^2(\mathbb{R}) : \widehat{f}(\omega) = 0, |\omega| > \sigma\}$.

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.05.013

The Shannon sampling theorem plays a fundamental role in data processing. It states that any $f \in B_\sigma$ can be reconstructed from its sampled values $\{f(x_k) : x_k := k\pi/\sigma, k \in \mathbb{Z}\}$ using the formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(x_k)\mathrm{sinc}(\sigma x/\pi - k), \quad x \in \mathbb{R},$$

where $\mathrm{sinc}(x) := \frac{\sin \pi x}{\pi x}$ for $x \neq 0$ and $\mathrm{sinc}(0) := 1$, and the series converges absolutely and uniformly on any finite interval of $\mathbb{R}$. See for example [1] and references therein.

Since only a finite number of samples can be used in practice, and the truncation error of the Shannon sampling formula is substantial due to the slow decay of the sinc function, it is expected that a modification of the sinc function by a multiplier will yield a better convergence rate of the Shannon sampling formula. A typical multiplier is the scaled Gaussian function $G_r(x) := \exp\left(-\frac{x^2}{2r^2}\right)$ for $x \in \mathbb{R}$ and $r > 0$. This leads to the localized sampling

$$(\mathcal{T}_h f)(x) := b \sum_{k\in\mathbb{Z}_m(x)} f(kh)\mathrm{sinc}[b(h^{-1}x - k)]G_r(h^{-1}x - k), \quad x \in \mathbb{R}, \tag{1}$$

where the oversampling factor $b \in (0, 1]$, the grid spacing $h > 0$, the truncation level $m \in \mathbb{N}$. Here we let $\mathbb{Z}_m(x) := \{k \in \mathbb{Z} : |[h^{-1}x] - k| \leq m\}$ and $[x]$ be the integer part of x.

The error bounds for approximating a band-limited function f and its derivatives by using (1) have been obtained [2,3]. Operator (1) has applications in digital signal processing and in obtaining numerical solutions of partial differential equations.

Since in applications the sampled values are obtained via real-world acquisition devices, we need to consider situations where the samples are not precise. See for example [1, pp. 84–96] and references therein. The goal of the present work is to study the localized sampling in the presence of noise,

$$(\mathcal{T}_h \tilde{f})(x) := b \sum_{k\in\mathbb{Z}_m(x)} \tilde{f}(kh)\mathrm{sinc}[b(h^{-1}x - k)]G_r(h^{-1}x - k), \quad x \in \mathbb{R}, \tag{2}$$

where $\{\tilde{f}(kh) : k \in \mathbb{Z}\}$ denotes the sampled values corrupted by noise. The level of corruption, for each given $h > 0$, can be measured by the quantity

$$\gamma(f) := \sup\{|\tilde{f}(kh) - f(kh)| : k \in \mathbb{Z}\}. \tag{3}$$

We will obtain error bounds for operator (2) for approximating a band-limited function and give its applications to various types of error.

2. Error estimates

The following result gives the error bounds concerning the approximation of a band-limited function f by using the localized sampling in the presence of noise.

Theorem 2.1. *If $f \in B_\sigma$, $b \in (0, 1]$, $h \in (0, b\pi/\sigma)$, $\mathcal{T}_h \tilde{f}$ is given in (2) and $\alpha := \min\{r(b\pi - h\sigma), (m-2)/r\} \geq 1$, then*

$$\|f - \mathcal{T}_h \tilde{f}\|_\infty \leq \beta e^{-\alpha^2/2}\|f\|_2 + b(2 + r)\gamma(f),$$

where $\beta := \frac{2\sqrt{\sigma\pi}}{\pi^2\alpha^2}(\sqrt{2\pi}\alpha + e^{\frac{3}{2r^2}})$ and $\gamma(f)$ is given in (3).

For the proof we need the following result concerning approximating a band-limited function f by using the localized sampling with exact samples.