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The motion of a viscous filament in a porous medium or Hele–Shaw cell: a physical realisation of the Cauchy–Riemann equations

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Abstract

We consider the motion of a thin filament of viscous fluid in a Hele–Shaw cell. The appropriate thin film analysis and use of Lagrangian variables leads to the Cauchy–Riemann system in a surprisingly direct way. We illustrate the inherent ill-posedness of these equations in various contexts.

Keywords: Cauchy–Riemann; Saffman–Taylor instability; Hele–Shaw; Viscous fingering; Unstable flow; Porous medium; Ill-posed; Filament; Displacement

1. Introduction

The displacement of a fluid by another of lower viscosity in a two-dimensional porous medium or a Hele–Shaw cell is subject to the Saffman–Taylor instability, which in practice often leads to what is known as viscous fingering. (An early, and key paper in this subject was that of Saffman and Taylor [5], which concerned the Hele–Shaw version of the problem, but the paper of Hill [3] even earlier had discussed the phenomenon in the context of miscible displacement in porous media. A review of the subject can be found in [4].) In this work, we relate this instability to the Cauchy–Riemann equations of

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complex analysis, for which the Cauchy problem is well known to be ill-posed. In particular, we show that the motion of a thin filament of viscous fluid between two effectively inviscid fluids gives a physical realisation of this Cauchy problem.

In the Hele–Shaw problem with two incompressible fluids, of which one is inviscid and the other has viscosity μ, the standard model leads to a moving boundary problem with Laplace's equation to be solved in the viscous fluid and constant pressure in the inviscid fluid. The pressure p and velocity $\mathbf{v}$ in the viscous fluid are, in suitable units, related by Darcy's law

$$\mathbf{v} = -\nabla p. \tag{1}$$

For incompressible flow we have

$$\nabla^2 p = 0$$

in the respective fluid regions. At interfaces separating the fluids, we assume the simple conditions

$$p = \text{constant} \tag{2}$$

and

$$-\frac{\partial p}{\partial n} = V_n, \tag{3}$$

where $\partial/\partial n$ is the derivative normal to the interface and V_n is its normal velocity. The effects of surface tension are ignored in these conditions. The model is completed by appropriate singularities representing the driving mechanism for the fluid motion, and by fixed boundary conditions as necessary.

The linear stability of a planar interface is obtained by routine analysis. Suppose the viscous fluid is to the right of a slightly perturbed planar interface $x = Vt + \epsilon\tilde{x}(y, t)$ and the inviscid one to its left; when $\tilde{x}(y, t) = \mathrm{e}^{\alpha t} \sin ky$ the result of a linear analysis about a travelling-wave solution is that

$$\alpha = V|k|. \tag{4}$$

An interface with $V > 0$ is therefore unstable if the less viscous fluid displaces the more viscous one, and the growth rate is proportional to the wavenumber. This dispersion relation is itself reminiscent of the Cauchy–Riemann equations. For example, suppose that we consider the Cauchy–Riemann system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for $y > 0$, $-\infty < x < \infty$, thinking of y as a time variable and of data for u and v as being given on $y = 0$. Solutions in which $u(x, y) = U_0 \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{\alpha y}$, $v(x, y) = V_0 \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{\alpha y}$ lead to the dispersion relation $\alpha^2 = k^2$ with catastrophic growth at large y, due to the root $\alpha = |k|$, similar to the large-time instability implied by (4) with $V = 1$, but also with one stable root $\alpha = -|k|$, corresponding to $V = -1$ in (4). In the next section we describe a physical realisation of this system. By using a thin filament of fluid with *two* free surfaces, we are able to have a flow which simultaneously has a stable free surface and an unstable one, and so is able to replicate the dispersion relation of the Cauchy–Riemann system; indeed, it replicates the Cauchy–Riemann system itself.

2. Motion of a filament in a porous medium or Hele–Shaw cell

Suppose that the Hele–Shaw cell or porous medium is divided into two parts by a thin filament of the viscous fluid, while the remainder is filled with the inviscid fluid, and that a pressure difference $\Delta P = 1$