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Applied Mathematics Letters 20 (2007) 123–130
Applied Mathematics Letters
www.elsevier.com/locate/aml

Inequalities for Stieltjes integrals with convex integrators and applications

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Received 22 September 2005; accepted 23 February 2006

Abstract

Inequalities for a Grüss type functional in terms of Stieltjes integrals with convex integrators are given. Applications to the Čebyšev functional are also provided.

Keywords: Stieltjes integral; Grüss inequality; Čebyšev inequality; Convex functions

1. Introduction

In [3], the authors have considered the following functional:

$$D(f;u) := \int_a^b f(x)\mathrm{d}u(x) - [u(b)-u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \tag{1.1}$$

provided that the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ and the Riemann integral $\int_a^b f(t)\mathrm{d}t$ exist.

In [3], the following result in estimating the above functional has been obtained:

Theorem 1. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)| \le L|x-y| \quad \text{for any } x, y \in [a,b] \quad (L>0) \tag{1.2}$$

and f is Riemann integrable on $[a,b]$.

If $m, M \in \mathbb{R}$ are such that

$$m \le f(x) \le M \quad \text{for any } x \in [a,b], \tag{1.3}$$

then we have the inequality

$$|D(f;u)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.02.027

In [2], the following result complementing the above has been obtained:

Theorem 2. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is of bounded variation on $[a, b]$ and f is Lipschitzian with the constant $K > 0$. Then we have*

$$|D(f; u)| \le \frac{1}{2} K(b - a) \bigvee_a^b (u). \tag{1.5}$$

The constant $\frac{1}{2}$ is sharp in the above sense.

For a function $u : [a, b] \to \mathbb{R}$, define the associated functions Φ, Γ and Δ by:

$$\Phi(t) := \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t), \quad t \in [a, b]; \tag{1.6}$$
$$\Gamma(t) := (t-a)[u(b) - u(t)] - (b-t)[u(t) - u(a)], \quad t \in [a, b]$$

and

$$\Delta(t) := \frac{u(b) - u(t)}{b-t} - \frac{u(t) - u(a)}{t-a}, \quad t \in (a, b).$$

In [1], the following subsequent bounds for the functional $D(f; u)$ have been pointed out:

Theorem 3. *Let $f, u : [a, b] \to \mathbb{R}$.*

(i) *If f is of bounded variation and u is continuous on $[a, b]$, then*

$$|D(f; u)| \le \begin{cases} \sup\limits_{t\in[a,b]} |\Phi(t)| \bigvee_a^b (f), \\ \dfrac{1}{b-a} \sup\limits_{t\in[a,b]} |\Gamma(t)| \bigvee_a^b (f), \\ \dfrac{1}{b-a} \sup\limits_{t\in(a,b)} [(t-a)(b-t)|\Delta(t)|] \bigvee_a^b (f). \end{cases} \tag{1.7}$$

(ii) *If f is L-Lipschitzian and u is Riemann integrable on $[a, b]$, then*

$$|D(f; u)| \le \begin{cases} L \displaystyle\int_a^b |\Phi(t)| \mathrm{d}t, \\ \dfrac{L}{b-a} \displaystyle\int_a^b |\Gamma(t)| \mathrm{d}t, \\ \dfrac{L}{b-a} \displaystyle\int_a^b (t-a)(b-t)|\Delta(t)| \mathrm{d}t. \end{cases} \tag{1.8}$$

(iii) *If f is monotonic nondecreasing on $[a, b]$ and u is continuous on $[a, b]$, then*

$$|D(f; u)| \le \begin{cases} \displaystyle\int_a^b |\Phi(t)| \mathrm{d}f(t), \\ \dfrac{1}{b-a} \displaystyle\int_a^b |\Gamma(t)| \mathrm{d}f(t), \\ \dfrac{1}{b-a} \displaystyle\int_a^b (t-a)(b-t)|\Delta(t)| \mathrm{d}f(t). \end{cases} \tag{1.9}$$

The case of monotonic integrators is incorporated in the following two theorems [1]: