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On the maximum area pentagon in a planar point set[☆]

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Abstract

A finite set of points in the plane is described as in convex position if it forms the set of vertices of a convex polygon. This work studies the ratio between the maximum area of convex pentagons with vertices in P and the area of the convex hull of P, where the planar point set P is in convex position.

Keywords: Convex position; Convex hull; Affine transformation

1. Introduction

Ref. [2] shows that in the study of motion-planning problems in robotics by using heuristics, the largest area polygons in a planar point set play an important role. Refs. [7,8] and [9] discuss these problems and contain the related results.

A finite set of points in the plane is described as in *convex position* if it forms the set of vertices of a convex polygon. Let P be a finite set of points in convex position in the plane; hence any subset of P is also a point set in convex position. Denote the area of the convex hull of $Q \subset P$ by $S(Q)$. For the sake of convenience we may call a subset $Q \subset P$ a polygon if Q forms the vertices of a polygon. Let

$$f_k(P) := \max\left\{\frac{S(Q)}{S(P)} : Q \subset P,\, P \text{ is in convex position}\right\}$$
$$f_k^{\text{conv}}(n) := \min\{f_k(P) : |P| = n,\ P \text{ is in convex position}\}.$$

Ref. [1] mainly studies $f_3^{\text{conv}}(n)$. In this work we evaluate $f_5^{\text{conv}}(n)$.

Instead of considering the ratio between the area of the convex hull of a point set and the area of the convex hull of its subset, [4–6] study the quantitative Steinitz Theorem and prove that any set whose convex hull contains a disk

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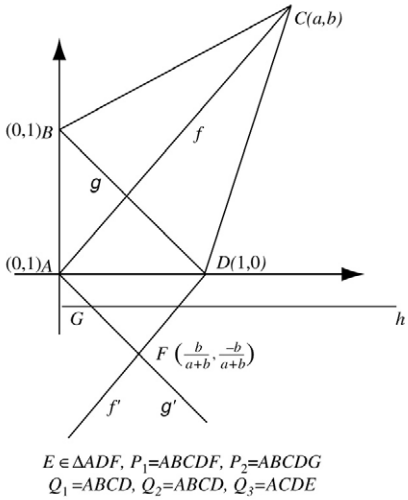

Fig. 1.

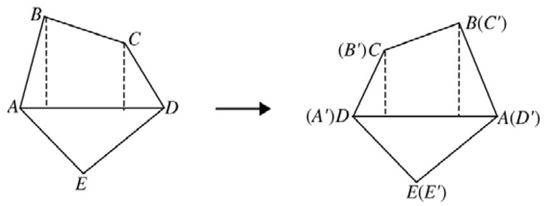

Fig. 2.

with center O and radius 1 has a subset of at most four points whose convex hull contains a disk with center O and radius $r_4 = \frac{\cos\frac{2\pi}{3}}{\cos\frac{\pi}{5}}$.

2. Main results

Lemma 1. $f_4^{\text{conv}}(5) = \frac{2}{5-\sqrt{5}}$.

Proof. Let P be a convex 5-gon with vertices A, B, C, D, E in clockwise order. Suppose the 4-gon $ABCD$ is a maximum area 4-gon in P. Given two triangles, there exists a unique affine transformation which transforms one triangle into another. So, without loss of generality we assume that $A = (0, 0), B = (0, 1), D = (1, 0), C = (a, b)$ $(a > 0, b > 0)$. Let $b \geq 1$; see Fig. 1. Indeed, when $b < 1$, the distance from B to the straight line AD is greater than the distance from C to the straight line AD, and we can reflect P about a vertical line, which does not change the ratio of the areas.

See Fig. 2. Relabel the vertices of P to ensure that the distance from C' to the straight line $A'D'$ is greater than the distance from B' to the straight line $A'D'$, and in this way we come to the case of $b \geq 1$.

Let Q_1, Q_2, Q_3 denote 4-gons $ABCD, ABDE, ACDE$ respectively. Let f be the line through A and C, and f' be the parallel line through D. Similarly, let g be the line through B and D, and g' be the parallel line through A. For Q_1 to be the maximum area 4-gon in P, E must lie completely above f' and g'. Define $F = f' \cap g'$; then $F = (\frac{b}{a+b}, \frac{-b}{a+b})$ and $E \in \triangle ADF$, and hence P is always contained in the convex 5-gon $P_1 = ABCDF$. Since $b \geq 1$, $S(Q_3) \geq S(Q_2)$; and since $S(Q_1) \geq S(Q_3)$, $S(\triangle ABC) \geq S(\triangle ADE)$. Suppose $E = (x_0, y_0)$,

$$S(\triangle ABC) = \frac{a}{2}, S(\triangle ADE) = \frac{-y_0}{2} \Rightarrow \frac{a}{2} \geq \frac{-y_0}{2} \Rightarrow y_0 \geq -a.$$

Then E lies above the horizontal line $h : y = -a$.