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Resolution of finite fuzzy relation equations based on strong pseudo-t-norms[☆]

Song-Chol Han[a,b], Hong-Xing Li[a,*], Jia-Yin Wang[a]

[a] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China*
[b] *Department of Mathematics and Mechanics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea*

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Abstract

This work studies the problem of solving a sup-T composite finite fuzzy relation equation, where T is an infinitely distributive strong pseudo-t-norm. A criterion for the equation to have a solution is given. It is proved that if the equation is solvable then its solution set is determined by the greatest solution and a finite number of minimal solutions. A necessary and sufficient condition for the equation to have a unique solution is obtained. Also an algorithm for finding the solution set of the equation is presented.

Keywords: Fuzzy relation equation; Strong pseudo-t-norm; Infinitely distributive strong pseudo-t-norm

1. Introduction

The resolution of fuzzy relation equations is one of the most important and widely studied problems in the field of fuzzy sets and fuzzy systems. The majority of fuzzy inference systems can be implemented by using the fuzzy relation equations [11]. Fuzzy relation equations can also be used for processes of compression/decompression of images and videos [8].

The sup–inf composite fuzzy relation equation was first proposed by Sanchez in 1976, and since then different kinds of fuzzy relation equations have been studied by many researchers [2–5,8–13,18]. Recently, Wang and Yu [15] introduced the notion of pseudo-t-norms. Building on this, Dai and Wang [1,14] considered the fuzzy relation equations with pseudo-t-norms. Meanwhile, Han and Li [7] introduced the concept of a strong pseudo-t-norm to correct some incorrect main results in [1,14–16].

In this work, we study in detail the resolution problem of a sup-T composite finite fuzzy relation equation, where T is an infinitely distributive strong pseudo-t-norm.

Throughout this work, L denotes the real unit interval $[0, 1]$ and J always stands for any nonempty set of subscripts.

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* Corresponding author.
E-mail address: lhxqx@bnu.edu.cn (H.-X. Li).

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2. Strong pseudo-t-norm

In this section, we give some definitions.

A binary operation T on L is called a pseudo-t-norm [15] if it satisfies the following conditions:

(T1) $T(1, a) = a$ and $T(0, a) = 0$ for all $a \in L$,
(T2) $a, b, c \in L$ and $b \leqslant c \Rightarrow T(a, b) \leqslant T(a, c)$.

A pseudo-t-norm T on L is said to be infinitely $\vee$-distributive [15] if it satisfies the following condition:

(T$_\vee$) $a, b_j \in L(j \in J) \Rightarrow T(a, \vee_{j\in J} b_j) = \vee_{j\in J} T(a, b_j)$.

A pseudo-t-norm T on L is said to be infinitely $\wedge$-distributive [16] if it satisfies the following condition:

(T$_\wedge$) $a, b_j \in L(j \in J) \Rightarrow T(a, \wedge_{j\in J} b_j) = \wedge_{j\in J} T(a, b_j)$.

A pseudo-t-norm T on L is said to be infinitely distributive [16] if it is both infinitely $\vee$-distributive and infinitely $\wedge$-distributive.

Let $A \in L^{L\times L}$. Define $I(A), T(A) \in L^{L\times L}$ as follows:

$$I(A)(a, b) := \vee\{u \in L \mid A(a, u) \leqslant b\},$$
$$T(A)(a, b) := \wedge\{u \in L \mid A(a, u) \geqslant b\},$$

where $a, b \in L$. It is tacitly assumed that $\vee\emptyset = 0$ and $\wedge\emptyset = 1$.

Theorem 2.1. *If T is an infinitely $\vee$-distributive pseudo-t-norm on L, then the following conditions are equivalent:*

(1) $T(a, c) \leqslant b \Leftrightarrow c \leqslant I(T)(a, b)$ *for all* $a, b, c \in L$;
(2) $T(a, 0) = 0$ *for all* $a \in L$.

Proof. (1) $\Rightarrow$ (2) For any $a \in L$, we have $0 \leqslant I(T)(a, 0)$. Using (1), we obtain $T(a, 0) \leqslant 0$, i.e., $T(a, 0) = 0$.

(2) $\Rightarrow$ (1) Let $a, b, c \in L$. If $T(a, c) \leqslant b$, then $I(T)(a, b) = \vee\{u \in L \mid T(a, u) \leqslant b\} \geqslant c$. Conversely, suppose $I(T)(a, b) \geqslant c$. Using (2), $0 \in \{u \in L \mid T(a, u) \leqslant b\} \neq \emptyset$. By (T2) and (T$_\vee$), $T(a, c) \leqslant T(a, I(T)(a, b)) = T(a, \vee\{u \in L \mid T(a, u) \leqslant b\}) = \vee\{T(a, u) \mid T(a, u) \leqslant b\} \leqslant b$. □

A pseudo-t-norm T on L is said to be strong [7] if it satisfies the following condition:

(T3) $T(a, 0) = 0$ for all $a \in L$.

It is obvious that t-norms and weak t-norms [6] are the particular cases of strong pseudo-t-norms. And there exists an infinitely $\vee$-distributive pseudo-t-norm that is not strong (see the pseudo-t-norm T_M in [15], for instance).

Example 2.1. Put

$$T(a, b) = \begin{cases} b, & a = 1, \\ 0, & a = 0, \\ 0, & 0 < a < 1, b = 0, \\ 1, & 0 < a < 1, b > 0, \end{cases}$$

where $a, b \in L$. Then T is an infinitely $\vee$-distributive strong pseudo-t-norm on L. However, it is not infinitely $\wedge$-distributive on L, since $T(a, \wedge\{b \in L \mid b > 0\}) = T(a, 0) = 0 \neq 1 = \wedge\{T(a, b) \mid b > 0\}$ for $0 < a < 1$.

Example 2.2 (*Wang and Yu [15]*). Let

$$T_W(a, b) = \begin{cases} b, & a = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b \in L$. Then T_W is an infinitely distributive strong pseudo-t-norm on L.