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Applied Mathematics Letters 19 (2006) 801–807

Applied Mathematics Letters

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On continuous dependence on coefficients of the Brinkman–Forchheimer equations

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Received 10 March 2004; received in revised form 27 October 2005; accepted 4 November 2005

Abstract

We prove continuous dependence of solutions of the Brinkman–Forchheimer equations on the Brinkman and Forchheimer coefficients in H^1 norm.

Keywords: Structural stability; Porous media; Continuous dependence on the coefficients

1. Introduction

In this work, we study the following initial-boundary value problem for the Brinkman–Forchheimer equations:

$$u_t = \gamma \Delta u - au - b|u|^\alpha u - \nabla p, \qquad \nabla \cdot u = 0, \qquad x \in \Omega, \qquad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u = 0, \qquad x \in \partial\Omega, \qquad t > 0. \tag{1.3}$$

Here $u = (u_1, u_2, u_3)$ is the fluid velocity vector, $\gamma > 0$ is the Brinkman coefficient, $a > 0$ is the Darcy coefficient, $b > 0$ is the Forchheimer coefficient, p is the pressure, $\alpha \in [1, 2]$ is a given number, Ω is a bounded domain of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is assumed to be of class C^2.

We study the problem of continuous dependence of solutions to the problem (1.1)–(1.3) on coefficients b and γ.

Continuous dependence of solutions on coefficients of equations is a type of structural stability, which reflects the effect of small changes in coefficients of equations on the solutions. Many results of this type can be found in the monograph of Ames and Straughan [1]. Structural stability in flows of fluid in porous media represented by the Darcy and Brinkman systems are investigated in the articles of Ames and Payne [2], Payne and Straughan [4–7]. In [7], Payne and Straughan considered the initial-boundary value problem (1.1)–(1.3) with $\alpha = 1$ which describes the flow of fluid in a saturated porous medium. They proved continuous dependence of solutions of the problem on the

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.11.002

coefficients b and γ in L^2 norm. Our aim is to show continuous dependence on these coefficients in a stronger norm, that is, in H^1 norm.

In the following we will use the function spaces $\tilde{H}_0^1(\Omega,\mathbb{R}^3) = \{u \in H_0^1(\Omega,\mathbb{R}^3) : \nabla\cdot u = 0\}$ and $\tilde{L}^2(\Omega,\mathbb{R}^3)$, where the latter space is the closure of $\tilde{H}_0^1(\Omega,\mathbb{R}^3)$ in $L^2(\Omega,\mathbb{R}^3)$. For simplicity we shall write $\tilde{L}^2(\Omega,\mathbb{R}^3) = \tilde{L}^2(\Omega)$ and $\tilde{H}_0^1(\Omega,\mathbb{R}^3) = \tilde{H}_0^1(\Omega)$. We use the notation $\|\cdot\|_p$ for the norm in $L^p(\Omega)$. We denote by $\|\cdot\|$ the norm and by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\Omega)$.

2. Continuous dependence on the Forchheimer coefficient

In this section, we prove that the solution of the problem (1.1)–(1.3) depends continuously on the Forchheimer coefficient b in $H^1(\Omega)$ norm.

Using the Faedo–Galerkin method we can prove the following existence and uniqueness theorem; see, for instance, [3, Theorem 9.3 and Theorem 10.2].

Theorem 1. *Assume that $1 \le \alpha \le 2$. Then for any $u_0 \in \tilde{H}_0^1(\Omega)$, there exists a unique solution $u \in C([0,T];\tilde{H}_0^1(\Omega))$ of the problem (1.1)–(1.3). Furthermore, we have*

$$\sup_{0\le t\le T}\|\nabla u(t)\| \le D \qquad \text{and} \qquad \int_0^T \|u_t(t)\|^2 \mathrm{d}t \le D \tag{2.1}$$

for any $T > 0$, where D is a generic positive constant depending on the initial data and the parameters of (1.1).

Let us show just how to get the estimates (2.1). First we multiply (1.1) by $u_t + u$ in $L^2(\Omega)$:

$$\begin{aligned} &2\|u_t(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma\|\nabla u(t)\|^2 + (a+1)\|u(t)\|^2 + \frac{2b}{\alpha+2}\int_\Omega |u(x,t)|^{\alpha+2}\mathrm{d}x\right] \\ &\quad + 2\gamma\|\nabla u(t)\|^2 + 2a\|u(t)\|^2 + 2b\int_\Omega |u(x,t)|^{\alpha+2}\mathrm{d}x = 0. \end{aligned} \tag{I_1}$$

It follows from this inequality that the function

$$\Phi(t) = \gamma\|\nabla u(t)\|^2 + (a+1)\|u(t)\|^2 + \frac{2b}{\alpha+2}\int_\Omega |u(x,t)|^{\alpha+2}\mathrm{d}x$$

satisfies the inequality $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \frac{2a}{a+1}\Phi(t) \le 0$. The latter implies the estimate

$$\gamma\|\nabla u(t)\|^2 + (a+1)\|u(t)\|^2 + \frac{2b}{\alpha+2}\int_\Omega |u(x,t)|^{\alpha+2}\mathrm{d}x \le D_1 \mathrm{e}^{-\frac{2a}{a+1}t}, \tag{I_2}$$

where

$$D_1 = \gamma\|\nabla u_0\|^2 + (a+1)\|u_0\|^2 + \frac{2b}{\alpha+2}\int_\Omega |u_0(x)|^{\alpha+2}\mathrm{d}x.$$

The boundedness of $\Phi(t)$ and hence the first of the estimates (2.1) follows from (I_2); the second of the estimates (2.1) follows if we integrate (I_1) over time and exploit the boundedness of $\Phi(t)$.

Now assume that (u, p) is the solution of the problem

$$\begin{aligned} &u_t = \gamma\Delta u - au - b_1|u|^\alpha u - \nabla p, \qquad \nabla\cdot u = 0, \qquad x\in\Omega, \qquad t>0, \\ &u(x,0) = u_0(x), \qquad x\in\Omega, \\ &u = 0, \qquad x\in\partial\Omega, \qquad t>0, \end{aligned}$$

and (v, q) is the solution of

$$\begin{aligned} &v_t = \gamma\Delta v - av - b_2|v|^\alpha v - \nabla q, \qquad \nabla\cdot v = 0, \qquad x\in\Omega, \qquad t>0, \\ &v(x,0) = u_0(x), \qquad x\in\Omega, \\ &v = 0, \qquad x\in\partial\Omega, \qquad t>0. \end{aligned}$$