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Invariance principles for hybrid systems with memory

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ABSTRACT

Hybrid systems with memory are dynamical systems exhibiting both delayed and hybrid dynamics. Such systems can be described by hybrid functional inclusions. Classical invariance principles play an instrumental role in proving stability and convergence of dynamical systems. Invariance principles for general hybrid systems with delays, however, remain an open topic. In this paper, we prove invariance principles for hybrid systems with memory, using both Lyapunov–Razumikhin function and Lyapunov–Krasovskii functional methods. These invariance principles are then applied to derive two stability results as corollaries.

1. Introduction

Hybrid systems with memory refer to a class of dynamical systems exhibiting both delayed and hybrid dynamics. Such systems naturally occur, e.g. in control applications where hybrid control algorithms are used with the presence of delays in the control loop, and have attracted considerable attention in the past decade (see, e.g., [1–8]). Classical invariance principles [9,10] play an instrumental role in proving stability and convergence properties of dynamical systems. As such, they have been extended by various authors to hybrid dynamical systems (e.g., [4,11–18]) and time-delay systems (e.g., [4,19,20]).

Invariance principles for general hybrid systems with delays, however, remain an open topic. The main difficulty in establishing such results lies in proving certain invariance properties of the limit set of memories resulted from a hybrid trajectory. For ordinary differential equations or hybrid systems without delays, the limit set involves no memory and is a subset of the Euclidean space. For delay differential equations, such memories are always encoded in continuous functions and the notion of uniform convergence topology is well-suited for studying their limits. The memory for hybrid systems with delays, however, could include discontinuities caused by jumps. Consequently, the uniform convergence topology is no longer appropriate for analyzing the limit behaviors of such systems.

Recent work in [21,22] introduces a framework for studying hybrid systems with delays through generalized solutions. To overcome the difficulty in handling discontinuities caused by jumps in hybrid systems with delays, the results in [21,22] rely on a phase space of hybrid memory arcs equipped with the graphical convergence topology, instead of a space of piecewise continuous functions equipped with the conventional uniform convergence topology. By using tools from functional differential inclusions, basic existence and well-posedness results have been established [23]. Moreover, stability results using both Lyapunov–Razumikhin function and Lyapunov–Krasovskii functional methods have also been proved [22,24].

The main contributions of this paper include two new invariance principles established for hybrid systems with memory. This work was made possible by applying the notion of graphical convergence of hybrid memory arcs along the line of work

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in [21,22]. As main applications of these new invariance principles, we show that two stability results for hybrid systems with memory can be derived as immediate corollaries.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries for hybrid systems with memory, mainly taken from [23], and prove a basic proposition that concludes weak invariance of the limit set, in graphical convergence sense, of the memories resulted from a hybrid trajectory. Section 3 include the main results on invariance principles and Section 4 illustrate two main applications of the invariance principles in the stability analysis of hybrid systems with memory. We illustrate the stability results with a simple example in Section 5 and conclude the paper in Section 6.

2. Preliminaries

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with its norm denoted by $|\cdot|$; $\mathbb{Z}$ denotes the set of all integers; $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{\leq 0} = (-\infty, 0]$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}_{\leq 0} = \{0, -1, -2, \ldots\}$.

2.1. Hybrid systems with memory

Definition 2.1 (*[23]*)**.** Consider a subset $E \subseteq \mathbb{R} \times \mathbb{Z}$ with $E = E_{\geq 0} \cup E_{\leq 0}$, where $E_{\geq 0} := (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \cap E$ and $E_{\leq 0} := (\mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0}) \cap E$. It is called a *compact hybrid time domain with memory* if

$$E_{\geq 0} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

and

$$E_{\leq 0} = \bigcup_{k=1}^{K} ([s_k, s_{k-1}], -k+1)$$

for some finite sequence of times $s_K \leq \cdots \leq s_1 \leq s_0 = 0 = t_0 \leq t_1 \leq \cdots \leq t_J$. It is called a *hybrid time domain with memory* if, for all $(T, J) \in E_{\geq 0}$ and all $(S, K) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $(E_{\geq 0} \cap ([0, T] \times \{0, 1, \ldots, J\})) \cup (E_{\leq 0} \cap ([-S, 0] \times \{-K, -K+1, \ldots, 0\}))$ is a compact hybrid time domain with memory. The set $E_{\leq 0}$ is called a *hybrid memory domain*.

Definition 2.2 (*[23]*)**.** A *hybrid arc with memory* consists of a hybrid time domain with memory, denoted by $\operatorname{dom} x$, and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous on $I^j = \{t : (t, j) \in \operatorname{dom} x\}$ for each $j \in \mathbb{Z}$ such that I^j has nonempty interior. In particular, a hybrid arc x with memory is called a *hybrid memory arc* if $\operatorname{dom} x \subseteq \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0}$. We shall simply use the term *hybrid arc* if we do not have to distinguish between the above two hybrid arcs. We write $\operatorname{dom}_{\geq 0}(x) := \operatorname{dom} x \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ and $\operatorname{dom}_{\leq 0}(x) := \operatorname{dom} x \cap (\mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0})$.

We shall use $\mathcal{M}$ to denote the collection of all hybrid memory arcs. Moreover, given $\Delta \in [0, \infty)$, we denote by $\mathcal{M}^\Delta$ the collection of hybrid memory arcs φ satisfying the following two conditions: (1) $s + k \geq -\Delta - 1$ for all $(s, k) \in \operatorname{dom} \varphi$; and (2) there exists $(s', k') \in \operatorname{dom} \varphi$ such that $s' + k' \leq -\Delta$.

Given a hybrid arc x, we define an operator $\mathcal{A}^\Delta_{[\cdot,\cdot]} x : \operatorname{dom}_{\geq 0}(x) \to \mathcal{M}^\Delta$ by

$$\mathcal{A}^\Delta_{[t,j]} x(s, k) = x(t + s, j + k),$$

for all $(s, k) \in \operatorname{dom}(\mathcal{A}^\Delta_{[t,j]} x)$, where

$$\operatorname{dom}(\mathcal{A}^\Delta_{[t,j]} x) := \Big\{(s, k) \in \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0} : (t + s, j + k) \in \operatorname{dom} x,\ s + k \geq -\Delta_{\inf}\Big\},$$

where

$$\Delta_{\inf} := \inf\Big\{\delta \geq \Delta : \exists (t + s, j + k) \in \operatorname{dom} x \text{ s.t. } s + k = -\delta\Big\}.$$

It follows that if $\mathcal{A}^\Delta_{[0,0]} x \in \mathcal{M}^\Delta$, then $\mathcal{A}^\Delta_{[t,j]} x \in \mathcal{M}^\Delta$ for any $(t, j) \in \operatorname{dom}_{\geq 0}(x)$.

Definition 2.3. A *hybrid system with memory of size* Δ is defined by a 4-tuple $\mathcal{H}^\Delta_\mathcal{M} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$:

- a set $\mathcal{C} \subseteq \mathcal{M}^\Delta$, called the *flow set*;
- a set-valued functional $\mathcal{F} : \mathcal{M}^\Delta \rightrightarrows \mathbb{R}^n$, called the *flow map*;
- a set $\mathcal{D} \subseteq \mathcal{M}^\Delta$, called the *jump set*;
- a set-valued functional $\mathcal{G} : \mathcal{M}^\Delta \rightrightarrows \mathbb{R}^n$, called the *jump map*.

Definition 2.4. A hybrid arc is a *solution to the hybrid system* $\mathcal{H}^\Delta_\mathcal{M}$ if $\mathcal{A}^\Delta_{[0,0]} x \in \mathcal{C} \cup \mathcal{D}$ and:

(S1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \operatorname{dom}_{\geq 0}(x)$,

$$\mathcal{A}^\Delta_{[t,j]} x \in \mathcal{C}, \qquad \dot{x}(t, j) \in \mathcal{F}(\mathcal{A}^\Delta_{[t,j]} x), \tag{2.1}$$