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# Change-of-bases abstractions for non-linear hybrid systems

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ABSTRACT

We present abstraction techniques that transform a given non-linear dynamical system into a linear system, or more generally, an algebraic system described by polynomials of bounded degree, so that invariant properties of the resulting abstraction can be used to infer invariants for the original system. The abstraction techniques rely on a change-of-bases transformation that associates each state variable of the abstract system with a function involving the state variables of the original system. We present conditions under which a given change-of-bases transformation for a non-linear system can define an abstraction. Furthermore, the techniques developed here apply to continuous systems defined by Ordinary Differential Equations (ODEs), discrete systems defined by transition systems and hybrid systems that combine continuous as well as discrete subsystems.

The techniques presented here allow us to discover, given a non-linear system, if a change-of-bases transformation involving degree-bounded polynomials yielding an algebraic abstraction exists. If so, our technique yields the resulting abstract system, as well. Our techniques enable the use of analysis techniques for linear systems to infer invariants for non-linear systems. We present preliminary evidence of the practical feasibility of our ideas using a prototype implementation.



## 1. Introduction

In this paper, we explore a class of abstractions for non-linear autonomous systems (continuous, discrete and hybrid systems) using *Change-of-Bases* (CoB) transformations. CoB transformations are obtained for a given system by expressing the dynamics of the system in terms of a new set of variables that relate to the original system variables by means of a transformation. Such a transformation is akin to studying the system under a new set of "bases". We derive conditions on the transformations such that (a) the CoB transformations also define an *autonomous system* and (b) the resulting system abstracts the original system: i.e., all invariants of the abstract system can be inverted to obtain invariants for the original system. Furthermore, we often seek abstract systems through CoB transformations whose dynamics are of a simpler form, more amenable to automatic verification techniques. For instance, it is possible to use CoB transformations that relate an ODE with non-linear right-hand sides to an affine ODE, or transformations that reduce the degree of a system with polynomial right-hand sides. If such transformations can be found, then safety analysis techniques over the simpler abstract system can be used to infer safety properties of the original system.

In this paper, we make two main contributions: (a) we define CoB transformations for continuous, discrete and hybrid systems and provide conditions under which a given transformation is valid; (b) we provide search techniques for finding CoB transformations that result in a polynomial system whose right-hand sides are degree limited by some limit $d \geq 1$. Specifically, the case $d = 1$ yields an affine abstraction; and (c) we provide experimental evidence of the application of our techniques to a variety of ordinary differential equations (ODEs) and discrete programs.

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The results in this paper extend our previously published results that appeared in Hybrid Systems: Computation and Control (HSCC) 2011 [1]. The contributions of this paper include (a) an extension from linearizing CoB transformations to degree-bounded polynomial CoB transformations, (b) extending the theory from purely continuous system to purely hybrid systems, and (c) an improved implementation that can handle hybrid systems with some evaluation results over this implementation. On the other hand, our previous work also included an extension of the theory to differential inequalities and iterative techniques over cones. These extensions are omitted here. Finally, our work here is restricted to autonomous ODEs and programs without external inputs. All uncertainty in our system is restricted to the initial set of states, which are not considered explicitly in the process of abstracting the dynamics, as detailed in Section 2.2. Bounded disturbances can potentially be handled in a planned extension of this paper that considers iterations over cones rather than vector spaces [1].

### *1.1. Motivating examples*

In this section, we motivate the techniques developed in this paper by means of a few illustrative examples involving purely continuous ODEs and purely discrete programs.

Our first example concerns a continuous system defined by a system of Ordinary Differential Equations (ODEs):

**Example 1.1.** Consider a continuous system over $\{x, y\}$: $\dot{x} = xy + 2x$, $\dot{y} = -\frac{1}{2}y^2 + 7y + 1$, with initial conditions given by the set $x \in [0, 1]$, $y \in [0, 1]$. Using the transformation $\alpha : (x, y) \mapsto (w_1, w_2, w_3)$ wherein $\alpha_1(x, y) = x$, $\alpha_2(x, y) = xy$ and $\alpha_3(x, y) = xy^2$, we find that the dynamics over $\vec{w}$ can be written as

$$\dot{w_1} = 2w_1 + w_2, \qquad \dot{w_2} = w_1 + 9w_2 + \frac{1}{2}w_3, \qquad \dot{w_3} = 2w_2 + 16w_3.$$

Its initial conditions are given by $w_1 \in [0, 1]$, $w_2 \in [0, 1]$, $w_3 \in [0, 1]$. We analyze the system using the TimePass tool as presented in our previous work [2] to obtain polyhedral invariants:

$$\begin{array}{l} -w_1 + 2w_2 \geq -1 \ \wedge \ w_3 \geq 0 \ \wedge \ w_2 \geq 0 \ \wedge \\ -16w_1 + 32w_2 - w_3 \geq -17 \ \wedge \ 32w_2 - w_3 \geq -1 \ \wedge \\ 2w_1 - 4w_2 + 17w_3 \geq -4 \ \wedge \ 286w_1 - 32w_2 + w_3 \geq -32 \ \wedge \\ \cdots . \end{array}$$

Substituting back, we can infer polynomial inequality invariants on the original system including,

$$\begin{array}{l} -x + 2xy \geq -1 \ \wedge \ xy^2 \geq 0 \ \wedge \ -16x + 32xy - xy^2 \geq -17 \\ x \geq 0 \ \wedge \ 2x - 4xy + 17xy^2 \geq -4 \ \wedge \ \cdots . \end{array}$$

Finally, we link the presence of affine change-of-bases abstractions with a set of conserved quantities (first integrals) of the original system (see Lemma 2.4 in page 21). Recall that a conserved quantity $f(\vec{x}, t)$ is a function whose value is constant over all trajectories: i.e., for all time trajectories of the system $f(\vec{x}(t), t) = f(\vec{x}(0), 0)$. For this example, we infer the following conserved quantity on the underlying non-linear system:

$$\begin{array}{l} \left(\frac{e^{-9t}}{51} + \frac{1}{102}\left(50 + 7\sqrt{51}\right)e^{\left(-9+\sqrt{51}\right)t} + \frac{1}{102}\left(50 - 7\sqrt{51}\right)e^{-\left(9+\sqrt{51}\right)t}\right) x \\ \quad + \left(-\frac{1}{102}e^{-9t-\left(9+\sqrt{51}\right)t}\begin{pmatrix} 7e^{9t} - \sqrt{51}e^{9t} - 14e^{\left(9+\sqrt{51}\right)t} \\ +7e^{9t+\left(-9+\sqrt{51}\right)t+\left(9+\sqrt{51}\right)t} \\ +\sqrt{51}e^{9t+\left(-9+\sqrt{51}\right)t+\left(9+\sqrt{51}\right)t} \end{pmatrix}\right) xy \\ \quad + \left(\frac{1}{204}e^{-9t-\left(9+\sqrt{51}\right)t}\left(e^{9t} - 2e^{\left(9+\sqrt{51}\right)t} + e^{9t+\left(-9+\sqrt{51}\right)t+\left(9+\sqrt{51}\right)t}\right)\right) xy^2. \end{array}$$

Finally, if $x(0) \neq 0$, the map $\alpha$ is invertible and therefore, the ODE above can be integrated.

Note that not every transformation yields a linear abstraction. In fact, most transformations will not define an abstraction. The conditions for an abstraction are discussed in Section 2. ▲

Next, we motivate our approach on purely discrete programs, showing how CoB transformations can linearize a discrete program with non-linear assignments, modeled by a *transition system* [3]. In turn, we show how invariants of the abstract linearized program can be transferred back.

**Example 1.2.** Fig. 1 shows an example proposed originally by Petter [4] that considers a program that sums up all squares from 1 to $K^2$ for some input $K \geq 0$. Consider a very simple change-of-bases transformation wherein we add a new variable "$y2$" that tracks the value of $y^2$ as the loop is executed. It is straightforward to write assignments for "$y2$" in terms of itself, $x$, $y$. Finally, the resulting program has affine guards and assignments, making it suitable for polyhedral abstract interpretation [5,6]. The polyhedral analysis yields linear invariants at the loop head and the function exit in terms of the variables $x$, $y$, $y2$.