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Regulated solutions for nonlinear measure driven equations

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ABSTRACT

We obtain the existence of regulated solutions for measure integral equations driven by a nondecreasing function, thus modeling a large class of hybrid systems (without any restriction on their Zeno behavior). By working with Kurzweil–Stieltjes integrals and making use of a measure of noncompactness, we are able to avoid Lipschitz-type assumptions. We finally present very useful particular cases and further applications of our result.

1. Introduction

For several decades already, it became obvious the imperativeness of modeling the evolution of systems where perturbations intervene in the continuous evolution. First, only a finite number of discrete perturbations were allowed [1] or, when their number is infinite, it was imposed that they accumulate only once, at the right (e.g. [2]).

We place ourselves in the setting of the theory of measure driven equations, studying such hybrid systems with a countably infinite number of perturbations in a finite interval of time, with no restrictions on their accumulation points (that are known as Zeno points, see [3, p. 78]).

What is more, we allow to the function governing the equation to be non-absolutely integrable, in which case only regulated solutions can be hoped to be obtained (see [4] for linear problems and [5,6] for the nonlinear case).

More precisely, applying a Darbo-type fixed point theorem given in [7], we obtain the existence of regulated solutions for the measure integral equation

$$x(t) = x^0 + (\mathrm{HLS}) \int_0^t f\,(s, x(s))\, dg(s)$$

driven by a nondecreasing function $g : [0, 1] \to \mathbb{R}$.

The key point of our approach is the use of Henstock–Stieltjes integration in parallel with the Hausdorff measure of noncompactness, combination that allows us to obtain the existence of regulated solutions without any assumption of Lipschitz-type.

We thus relax the conditions imposed in Theorem 5.3 in [5] and Theorem 6.1 in [6], that focus on measure retarded integral equations in the same framework, i.e. g nondecreasing and left-continuous, involving Kurzweil–Henstock integrals.

Our problem is related to the measure driven differential problem

$$\begin{aligned} dx &= f\,(t, x)\, dg \\ x(0) &= x^0 \end{aligned}$$

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where dx and dg denote the distributional derivatives of the solution and the function g, respectively. We have to note that, unlike in the classical case $g(s) = s$, the equivalence between the sets of solutions of the two problems is a delicate matter. It relies on the chosen definition of solution and there are examples showing that an error in this direction can conduct to paradoxes (we refer to [8]).

The various definitions of solution concept for hybrid systems (see [9] or [10]) usually coincide when the system is Zeno-free, i.e. the moments of perturbation of continuous evolution does not accumulate in a finite amount of time. But they bring us to really different sets of solutions when such Zeno points occur. On the other hand, even if a supposition of lake of such Zeno behavior would be very convenient, many examples (as the system consisting of three balls where inelastic impacts are modeled by succession of simple impacts in [11], the bouncing ball in [12] or those described in [9]) certify the necessity of allowing this kind of behavior if we want to be able to analyze complex hybrid systems.

2. Notations and preliminary facts

Let X be a separable Banach space with norm $\|\cdot\|$ and corresponding distance d. Recall that a function $F : [0, 1] \to X$ is said to be regulated if there exist the limits $F(t+)$ and $F(s-)$ for every points $t \in [0, 1)$ and $s \in (0, 1]$. It is well-known [13], that the set of discontinuities of a regulated function is at most countable and that the space $G([0, 1], X)$ of regulated functions is a Banach space when endowed with the usual norm $\|F\|_C = \sup_{t\in[0,1]} \|F(t)\|$.

We will make use of the theory of Kurzweil–Stieltjes integration in Banach spaces, which is a particular case of Kurzweil integration [14].

Let $g : [0, 1] \to \mathbb{R}$. A partition of $[0, 1)$ is a finite collection of pairs $\{(I_i, \xi_i) : i = 1, \dots, p\}$, where $I_1, \dots, I_p$ are non-overlapping subintervals of $[0, 1)$, $\xi_i \in I_i$, $i = 1, \dots, p$ and $\cup_{i=1}^p I_i = [0, 1)$. A gauge δ on $[0, 1)$ is a positive function on $[0, 1)$. For a given gauge δ we say that a partition $\{(I_i, \xi_i) : i = 1, \dots, p\}$ is δ-fine if $I_i \subset [\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$, $i = 1, \dots, p$.

Definition 1. A function $f : [0, 1] \to X$ is said to be Henstock–Lebesgue–Stieltjes-integrable with respect to $g : [0, 1] \to \mathbb{R}$ on $[0, 1)$ (shortly, HL–Stieltjes integrable) if there exists a function denoted by (HLS) $\int_0^{\cdot} : [0, 1] \to X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε on $[0, 1)$ with

$$\sum_{i=1}^{p} \left\| f(\xi_i)(g(t_i) - g(t_{i-1})) - \left((\mathrm{HLS}) \int_0^{t_i} f(s)dg(s) - (\mathrm{HLS}) \int_0^{t_{i-1}} f(s)dg(s) \right) \right\| < \varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i), \xi_i) : i = 1, \dots, p\}$ of $[0, 1)$.

The HL–Stieltjes integrability is preserved on all sub-intervals of $[0, 1)$. The function $t \mapsto$ (HLS) $\int_0^t f(s)dg(s)$ is called the HL–Stieltjes primitive of f w.r.t. g on $[0, 1)$ (we refer to [4] or [15] for finite dimensional space X).

Remark 2. When $g(s) = s$, this definition gives the concept of HL-integrable function [16]. If moreover X is finite dimensional, in the preceding definition the norm can be put outside the sum, giving the equivalent concept of Henstock integral (see [16,17] or [18] for a comparison between the two notions in general Banach spaces).

As the HL–Stieltjes integral satisfies the Saks–Henstock Lemma (Lemma 1.13 in [15]), the proof of Theorem 1.16 in [15] works in our setting and gives:

Proposition 3. *Let $g : [0, 1] \to \mathbb{R}$ and $f : [0, 1] \to X$ be HL–Stieltjes integrable w.r.t. g. If g is regulated, then so is the primitive $h : [0, 1] \to X$, $h(t) =$ (HLS) $\int_0^t f(s)dg(s)$ and for every $t \in [0, 1]$,*

$$h(t^+) - h(t) = f(t)\left[g(t^+) - g(t)\right] \quad \text{and} \quad h(t) - h(t^-) = f(t)\left[g(t) - g(t^-)\right].$$

The space of all functions that are HL–Stieltjes-integrable w.r.t. g is denoted by $\mathcal{HLS}(g)$ and is endowed with the supremum norm of its primitive (that is regulated, see Proposition 3), namely the Alexiewicz norm w.r.t. g:

$$\|f\|_A^g = \sup_{t\in[0,1]} \left\| (\mathrm{HLS}) \int_0^t f(s)dg(s) \right\|.$$

For basic properties of Lebesgue–Stieltjes integral (that will also be used through the paper) we refer the reader to [19].

An important element in our work is the Hausdorff measure of noncompactness, which we recall now. For any bounded subset E of a metric space, let $\alpha(E)$ be the Hausdorff measure of non-compactness of E: the infimum of all $r > 0$ for which there exists a finite number of balls covering E, of radius smaller than r (see [20]). The following result offers a method to calculate the measure of noncompactness.

Proposition 4 ([21, Proposition 1.4]). *Let X be a separable Banach space and $(X_q)_q$ an increasing sequence of finite dimensional subspaces with $X = \overline{\cup_{q\in\mathbb{N}} X_q}$. Then for every bounded countable set $M = (a_m)_m \subset X$,*

$$\alpha(M) = \lim_{q\to\infty} \overline{\lim_{m\to\infty}} d(a_m, X_q).$$

We present in the sequel several basic properties of α in the framework of Stieltjes-type integrals.