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# Optimal harvesting for a stochastic regime-switching logistic diffusion system with jumps

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ABSTRACT

The optimization problem of fishing for a stochastic logistic model is studied in this paper. Besides a standard geometric Brownian motion, another two driving processes are taken into account: a stationary Poisson point process and a continuous-time finite-state Markov chain. The classical harvesting problem for this model is a big difficult puzzle since the corresponding Fokker–Planck equations with three types of noise are very difficult to solve. Our main goal of this paper is to work out the optimization problem with respect to stationary probability density. One of the main contributions is to provide a new equivalent approach to overcome this problem. More precisely, an ergodic method is used to show the almost surely equivalency between the time averaging yield and sustainable yield. Results show that the optimal strategy changes with environment. An interesting thing is that the optimal strategy for each state is equivalent to the global optimality.



## 1. Introduction

Investigations on various logistic-type systems are one of the most important themes in mathematical ecology. Optimal harvesting problem is an important and interesting topic from both biological and mathematical point of view. The optimal harvesting problems have received a lot of attention since *Clark*'s [1,2] classical works on the following deterministic logistic system with catch-per-unit-effort (CPUE) hypothesis:

$$\mathrm{d}X(t) = X(t)\,(a - bX(t))\,\mathrm{d}t - EX(t)\mathrm{d}t. \tag{1.1}$$

In Eq. (1.1), $X(t)$ represents the size of population at time $t$, $a$ denotes the intrinsic growth rate of $X(t)$, $a/b$ is the carrying capacity of the environment, $EX(t)$ is the CPUE hypothesis of harvesting function. *Clark* showed the optimal harvesting effort for Eq. (1.1) is $E^* = a/2$ and the maximum sustainable yield is $a^2/(4b)$. The fact that population systems are inevitably subject to various environmental noises is accepted by a large number of scholars, such as [3–8] and references cited there in. *Beddington* and *May* [9] showed the optimal harvesting for natural populations in a randomly fluctuating environment in *Science*. In their paper, they proved the optimal harvesting effort for

$$\mathrm{d}X(t) = X(t)\,(a - bX(t))\,\mathrm{d}t - EX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t) \tag{1.2}$$

is $(a - \sigma^2/2)/2$ and the maximum sustainable yield is $(a - \sigma^2/2)^2/(4b)$. Here, $B(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, F, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. $\sigma^2/2$ is the intensity of white noise. In addition, population systems may be affected by another type of environmental noise, namely, color noise or say telegraph noise. This telegraph noise can be illustrated as a switching between two or more sub-regimes of different environments [10,11]. For instance, the growth rate

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for some fish in dry season will be much different from it in rainy season. *Wang* [12] considered color noise on a stochastic logistic model with CPUE hypothesis. He studied the following system

$$\mathrm{d}X(t) = X(t)\left(a\left(r\left(t\right)\right) - bX\left(t\right)\right)\mathrm{d}t - EX\left(t\right)\mathrm{d}t + \sigma\left(r\left(t\right)\right)X\left(t\right)\mathrm{d}B(t), \tag{1.3}$$

where $r(t)$ is a continuous-time finite-state Markov chain with values in the finite state space $S = \{1, 2, \ldots, N\}$. For any state $i \in S$, $a(i)$ and $\sigma(i)$ are both positive constants. That is to say, Eq. (1.3) can be considered as a switching between $N$ regimes of Eq. (1.2). For each regime $i \in S$, it has different parameters. Results show that the optimal harvesting effort for Eq. (1.3) is $\sum_{i=1}^{N}\pi_i\left(a\left(i\right) - \sigma^2\left(i\right)/2\right)/2$ and the maximum time averaging yield is $\left[\sum_{i=1}^{N}\pi_i\left(a\left(i\right) - \sigma^2\left(i\right)/2\right)\right]^2/(4b)$, where $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ is the stationary distribution for the color noise $r(t)$. For Eq. (1.3), the author only obtained the optimization in time averaging sense, but did not get the classical optimal harvesting problem with sustainable yield function.

Anything else, population dynamics may suffer sudden environmental shocks, such as: earthquakes, epidemics, floods, toxic pollutants, hurricanes and so on. e.g., Tangshan Earthquake (China) in 1976, the Three Mile Island nuclear accident of waste furnace (US) in 1979, the Severe Acute Respiratory Syndromes emerged in 2003, the earthquake and the corresponding nuclear threat to Japan in 2011 and so on. These sudden environmental shocks will cause jumps in population size, and model (1.3) cannot describe these phenomena exactly. To describe this type of environmental noise, scholars introduce a jumping process into the underlying population dynamics, and use stochastic differential equations (SDEs) driven by jumping processes to explain these phenomena. We regard this type of noise as jumping noise. Rong Situ [13] (see page 32) said "stochastic perturbation of jumps in a dynamical system usually can be modeled as a stochastic integral with respect to some point processes, i.e. its martingale measure or counting measure". Motivated by this, we will use a stationary Poisson point process as the driven jumping processes in this paper. Let $p\left(t, \omega\right)$ represent a stationary $\mathcal{F}_t$-adapted Poisson point process. $N$ is the Poisson counting measure generated by $p\left(t, \omega\right)$. $\lambda$ is the intensity measure of $N$ which is defined on a finite measurable subset $\mathbb{Y} \subset (0, \infty)$. $\widetilde{N}$ is the compensated random measure defined by $\widetilde{N}\left(\mathrm{d}t, \mathrm{d}u\right) = N\left(\mathrm{d}t, \mathrm{d}u\right) - \lambda\left(\mathrm{d}u\right)\mathrm{d}t$. Consequently, the corresponding stochastic logistic hybrid jump-diffusion process driven by $p\left(t, \omega\right)$ has the following form:

$$\begin{aligned}\mathrm{d}X(t) &= X(t)\left(a\left(r\left(t\right)\right) - b\left(r\left(t\right)\right)X\left(t\right)\right)\mathrm{d}t - E\left(r\left(t\right)\right)X\left(t\right)\mathrm{d}t \\ &\quad + \sigma\left(r\left(t\right)\right)X\left(t\right)\mathrm{d}B(t) + X\left(t^-\right)\int_{\mathbb{Y}}\gamma\left(r\left(t\right), u\right)\widetilde{N}\left(\mathrm{d}t, \mathrm{d}u\right).\end{aligned} \tag{1.4}$$

Here, $X\left(t^-\right)$ is the left limit of $X\left(t\right)$. $\gamma\left(r(t), u\right)$ reflects the birth rate $(\gamma > 0)$ or death rate $(\gamma < 0)$ caused by the jumps to some extent. So, for each $i \in S$, we ask for $\gamma\left(i, u\right)$ to be a bounded function such that $\gamma\left(r\left(t\right), u\right) > -1$, $u \in \mathbb{Y}$. For more information, please see paper [14]. The other parameters have the same meanings as models (1.1)–(1.3). Brownian motion $B(t)$, Markov chain $r(t)$ and stationary Poisson point process $p\left(t, \omega\right)$ are always assumed to be mutually independent and defined on the same complete probability space $(\Omega, F, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Markov chain $r(t)$ is assumed to be irreducible since this standing hypothesis implies the hybrid system will switch from any sub-state to any other sub-state. In addition, this assumption also means that Markov chain has a unique stationary distribution [15] $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in R^{1\times N}$. Eq. (1.4) can be regarded as switchings from one sub-system to another according to Markov Chain. The switching between these $N$ sub-regimes is governed by the Markov chain on $S = \{1, 2, \ldots, N\}$.

The main contributions of this paper are listed as follows:

- the model includes three types of environmental noise which is more grounded in reality. The explicit solution for a logistic model with three types of environmental noise is given in Lemma 3.1;
- we provide a technique to handle the case when all the parameters in Eq. (1.4) are assumed to depend on the Markov chain $r(t)$ (see the proof of Theorem 3.1). This assumption is more reasonable than model (1.3) since all the parameters will change with the environment. Take harvesting effort as an example, $E(r(t))$ implies the harvesting effort will change with the environment;
- a new approach is given to do the optimal harvesting problems which can avoid the puzzle of solving the complicated Fokker–Planck equations;
- we give the numerical simulations for the stochastic logistic model with three types of noises, and give a brief introduction of its principle (see Section 6).

As far as we know, there are few results about Eq. (1.4), and this is the first attempt to study the optimal harvesting problems for Eq. (1.4).

## 2. Problem statement

For any twice continuously differentiable function $V(\cdot, i)$, the infinitesimal generator operator $\mathcal{L}_t$ for Eq. (1.4) is defined by

$$\begin{aligned}\mathcal{L}_t V(x, i) &:= V_x(x, i)x\left[a(i) - E(i) - b(i)x\right] + \frac{1}{2}V_{xx}(x, i)\sigma^2(i)x^2 + \sum_{j=1}^{N}\gamma_{ij}V(x, j) \\ &\quad + \int_{\mathbb{Y}}\left[V\left(x + \gamma(i, u)x\right) - V(x, i) - V_x(x, i)x\gamma(i, u)\right]\lambda\left(\mathrm{d}u\right), \quad i = 1, 2, \ldots, N,\end{aligned} \tag{2.1}$$