Nonlinear Analysis: Hybrid Systems 4 (2010) 432–440

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Counting paths on a chessboard with a barrier[☆]

Marcellino Gaudenzi

Dipartimento di Finanza dell'Impresa e dei Mercati Finanziari, Via Tomadini 30/A, Universitá di Udine, 33100 Udine, Italy

ARTICLE INFO

Article history:
Received 19 October 2009
Accepted 2 November 2009

Keywords:
Paths
Binomial trees
Barriers
Counting formulas
Parisian options

ABSTRACT

We are dealing with the problem of counting the paths joining two points of a chessboard in the presence of a barrier. The formula for counting all the paths joining two distinct positions on the chessboard lying always over a barrier is well known (see for example Feller (1968) [1], Kreher and Stinson (1999) [3]). The problem is here extended to the calculation of all the possible paths of n movements which stay exactly k times, $0 \leq k \leq n+1$, over the barrier. Such a problem, motivated by the study of financial options of Parisian type, is completely solved by virtue of five different formulas depending on the initial and final positions and on the level of the barrier.

1. Introduction

In the context of mathematical finance, in particular in the discrete methods for pricing financial derivatives, it is often important to count the number of paths joining two positions on a tree. There are several different cases of interest, especially when there is a barrier. So it is important to evaluate the number of all the possible paths on the tree, joining two given positions, lying always over a barrier which is set at a level higher than the initial and the final position. This problem is well known and admits a simple counting formula which is given by the difference of two binomial coefficients and which can be obtained by the "reflection principle" (see [1]). This issue is strictly connected to the ballot problem (see [2]) and the Catalan numbers (see [3]).

Motivating by the study of Parisian options (see [4,5]) the author met a generalization of this problem consisting in counting the number of all the possible paths joining two positions at the same level and staying not more than k times over a given barrier. In fact, for pricing Parisian options, it is necessary to compute the time that the underlying asset has spent over the barrier, either consecutively or cumulatively. Discretizing this problem through binomial trees, we need to count the number of all the trajectories on the tree lying a given number of times over the barrier. This number of times has to be counted either consecutively or cumulatively. In order to treat the former case it is sufficient to use binomial formulas which are already known, but in the latter case the known approach is no longer applicable.

In this paper this problem is presented in a general form. We use a chessboard in order to describe the possible paths using a well known situation. In fact on the chessboard a pawn can move only to consecutive positions which lie either over or under the previous position and this is exactly what occurs in the case of the paths on a binomial tree. So we consider the trajectory of a pawn which joins two positions. In order to obtain the maximal generality of the results, we remove the assumption that the initial and the final position lie on the same level. The barrier can be set at every possible level as well; in fact the assumption that the barrier lies over the initial and final positions will be a restriction of the problem in our case. So our main result (Theorem 1) consists in establishing a formula which provides the exact number of all the possible paths of n movements joining two arbitrary positions of the chessboard and lying exactly k times $(0 \leq k \leq n+1)$ over a barrier set at an arbitrary level l. More precisely, 'over' means larger than or equal to, and both the initial and the final position are

☆ This research was supported by MIUR (Prin 2007).
E-mail address: marcellino.gaudenzi@uniud.it.

1751-570X/$ – see front matter
doi:10.1016/j.nahs.2009.11.001

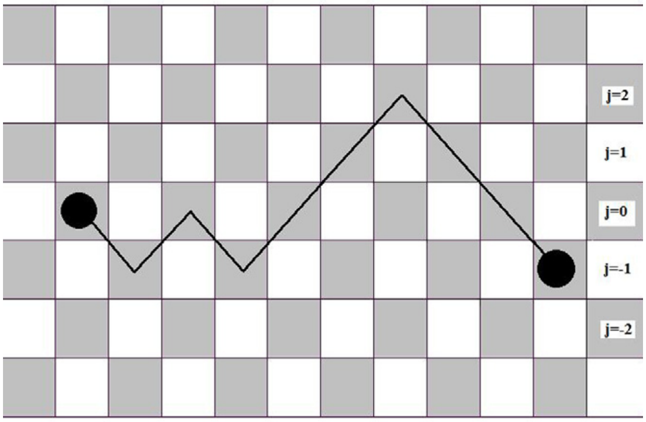

Fig. 1. A path of the pawn in the case $n = 9, r = 4$. The path stays 10 times ($k = 10$) over the level -1 ($l = -1$), 7 times over the level 0, 3 times over the level 1, 1 time over the level 2.

counted as well (if they stay over the barrier). As an immediate consequence, we obtain the number of paths lying not more than k times over the barrier (Corollary 2).

The counting formula requires us to consider five different cases, which depend on the choice of the parameters. In particular, when the barrier lies between the initial and final positions, the formula is heavier. The results obtained allow us to reduce the computational complexity of the routines needed for pricing Parisian options using binomial trees from $O(n^3)$ to $O(n^2)$.

2. A chessboard problem

The main question analyzed in this paper consists of the following problem formulated on an unbounded chessboard.

Problem 1. A pawn moves on a chessboard from a fixed initial position by making n, $n \geq 1$, right movements: r up and $n - r$ down. The initial position is set as vertical level 0; therefore the vertical level reached by the pawn at the end of the path is $j = 2r - n$. A barrier, at level l, $-n \leq l \leq n$, is considered. How many paths of the pawn lie exactly k times over the barrier?

In order to count the number of times that the path stays over the level l the initial and the final positions are counted as well; hence $0 \leq k \leq n + 1$ (see Fig. 1). So $k = n + 1$ means that the path stays always over the barrier. Throughout the paper 'over the barrier' means always the weak inequality, i.e. the equality is allowed.

Consider first the case $k = n + 1$. It is well known (see [1]) that the number of paths of the pawn which stay always over the level l is given by the following formula:

$$b^l_{n,r} := \begin{cases} \binom{n}{r} - \binom{n}{r+1-l} & \text{if } l \leq \min\{0, 2r - n\} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here we assume $\binom{n}{k} = 0$ if $n < k$. Clearly if $l > \min\{0, 2r - n\}$ there are no paths staying always over the barrier, in fact every path starts at level 0 and arrives at level $j = 2r - n$. We also remark that if $n > r + 1 - l$ then all the paths joining the initial and the final position stay always over the barrier.

In the particular case $l = 0$ and $2r - n \geq 0$ we get

$$b_{n,r} := b^0_{n,r} = \binom{n}{r} - \binom{n}{r+1} = \frac{2r - n + 1}{r + 1}\binom{n}{r}. \tag{2}$$

Such a number counts all the trajectories lying always over the initial position and arriving at level $j = 2r - n \geq 0$. In the sequel the numbers $b_{n,r}$ will play a special role; in fact our results are based on such numbers. However, the statement of the results becomes simpler if we replace the dependence on the number r of up movements with the level of arrival, $j = 2r - n$, of the path. Hence we set

$$B_{n,j} := \begin{cases} b_{n,\frac{j+n}{2}} & \text{if } n + j \text{ is even} \\ 0 & \text{if } n + j \text{ is odd} \end{cases}$$