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Chaotic behavior analysis based on corner bifurcations

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ABSTRACT

In this paper, a mathematical analysis in order to generate a chaotic behavior for bounded piecewise smooth systems of dimension three submitted to one of its specific bifurcations, namely the corner one, is proposed. This study is based on period doubling method.

1. Introduction

Piecewise Smooth Systems (P.W.S) are dynamic systems given by two or more sets of differential equations, such that those systems switch from one phase space (defined by smooth dynamics) to another one when some switching conditions are satisfied. Those regions are separated by finitely smooth codimension one boundaries; in fact, discontinuities represent an important class of nonlinear dynamical systems. These systems exhibit complex behavior that cannot be explained only by using the bifurcation analysis associated to smooth dynamic systems [1]. They can undergo all the bifurcations related to smooth systems but they also have specific type of bifurcations called border collision bifurcations, as grazing, sliding, chattering, corner... that occur when a fixed point collides with a borderline resulting in a sign modification of the eigenvalues associated to the jacobian matrix. Note that many papers deal with piecewise smooth maps that are continuous across the borderlines [2,3]. However, many switching dynamic systems as DC–DC converters [4], thyristor controlled reactor circuits [5] and digitally controlled systems [6] give rise to discontinuous maps. The bifurcation theory for one and two dimensional continuous piecewise maps has been developed in [2,7–10]. M. Feigin obtained some results on periodic orbits for continuous maps [11] and some authors extend Feigin's approach to the case of one dimensional discontinuous maps [12]: the idea is to consider a piecewise linear approximation of the related system in the neighborhood of the discontinuity plane in order to obtain general existence conditions of period-1, period -2 fixed points before and after a border collision bifurcation. On the other hand, one of the most striking features of piecewise smooth systems is that they exhibit sudden transitions from periodic attractors to chaos in the absence of any period doubling or other bifurcation cascades usually observed in smooth systems [13]. Hereafter, after treating the case of way to chaos: via grazing bifurcation in [14] and via the sliding one in [15], we consider the case of way to chaos via corner bifurcation: it is certainly the easier one to understand graphically, but also one of the most tedious to analyze. This is due to its specific topological structure (the discontinuity boundary is a corner -type formed by the intersection between two smooth codimension one surfaces

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at a nonzero angle) and to its very particular assumptions. The paper provides the foundations of a rational approach of transition to chaos by period doubling base on the related Poincaré map introduced by di Bernardo and co-workers [8].
On the other hand, an important feature to consider is that there exist two types of chaotic attractors: the fragile chaos corresponding to the disappearance of the attractors or the coexistence of another attractor when some parameter is perturbed, and the robust chaos corresponding to the absence of periodic windows that could destroy the chaotic behavior of the system for small perturbations of the bifurcation parameter. In many practical applications, it is of the utmost importance to maintain a robust chaotic comportment as in communications, electrical engineering [16,17], so, an overview of some important issues concerning the robustness of chaos in smooth dynamic systems is given in [18–20]. The result obtained in [18] contradicts the conjecture that robust chaos cannot occur for smooth systems. Here, the analysis concerns piecewise smooth systems submitted to corner bifurcations. Our approach guarantees a robust chaotic behavior (in the meaning that for some small bifurcation parameter variations, the system behavior stays chaotic), and the robustness analysis is different from those given in the previous results and is obtained directly from the proposed approach.

The paper is organized as follows: In Section 2, some definitions on the Poincaré map associated to the corner bifurcations are given and the problem statement is established; in Section 3, the periodic existence and uniqueness problem solution's analysis is presented based on the Implicit Functions theorem; a way to chaos by period doubling method is proposed in Section 4. Finally, an example based on this approach with simulations results and comments is presented in Section 5.

2. Definitions and problem statement

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x) & \text{if } H_1(x) \geq 0 \text{ or } H_2(x) \geq 0 \\ F_2(x) & \text{if } H_1(x) < 0 \text{ and } H_2(x) < 0 \end{cases} \tag{1}$$

where $x : I \longrightarrow D$, D is an open bounded domain of R^n with $n \geq 3$, I is the time interval.

$$F_1, F_2 : C_{abs}(I, D) \longrightarrow C^k(I, D), \quad k \geq 2.$$

where $C^k(I, D)$ is the set of C^k functions defined on I and having values in R^n, the norm for $C^k(I, D)$ is defined as follows:

$$x \in C^k(I, D) : \|x\|_k = \sup_{t\in I} \|x(t)\|_n + \sup_{t\in I} \|\dot{x}(t)\|_n + \cdots + \sup_{t\in I} \left\|x^{(k)}(t)\right\|_n$$

$x^{(k)}(.)$ is the kth derivative of $x(.)$, $\|.\|_n$ is a norm defined on R^n and $C_{abs}(I, D)$ is the set of absolutely continuous functions defined on I and having values in D provided with the norm: $x \in C_{abs}(I, D) : \|x\| = \sup_{t\in I} \|x(t)\|_n$

According to [21], $(C^k(I, D), \|.\|)$ and $(C_{abs}(I, D), \|.\|)$ are Banach spaces.

Now, let us consider for $i = 1, 2$ two C^1 applications: $H_i : D \longrightarrow R$ such that they define the sets:

$$S_1^0 = \{x(t) \in D : H_1(x(t)) = 0\} \quad \text{and} \quad S_2^0 = \{x(t) \in D : H_2(x(t)) = 0\}.$$

It is assumed that S_1^0 and S_2^0 intersect along a corner (noted C) with smooth codimension two surfaces. This generates a non-zero angle. Now let us consider (without loss of generality) an initial point $(x_0, t_0) = (0, 0)$ (after an eventual translation), such that $(0, 0)$ is on the corner C, i.e. satisfies the following assumption:

Assumption:

(H-1) $\nabla H_1(0, 0) \times \nabla H_2(0, 0) \neq 0$.

Moreover, the following regions are not empty:

$$D_{int} = \{x(t) \in D : H_1(x(t)) < 0 \quad \text{and} \quad H_2(x(t)) < 0\}$$
$$D_{out} = \{x(t) \in D : H_1(x(t)) \geq 0 \quad \text{or} \quad H_2(x(t)) \geq 0\}.$$

Both vector fields F_1 and F_2 are defined on both sides of D_{int} and D_{out} respectively.

Now, in order to define the corner-collision trajectory, two possibilities occur: external and internal collision, where, for the sake of clarity, both collisions are given for systems of dimension 2 (see Fig. 1):

Remark 1. The internal corner becomes an external one by changing the corresponding obtuse angle to an acute one. So, the complete study of the internal case (for example) is sufficient.

So, in order to avoid sliding phenomena, one requires the following assumptions for each case:

Assumptions:

For the external corner collision:

(A-1) $\langle F_i(0, 0), \nabla H_1(0, 0)\rangle < 0$ and $\langle F_i(0, 0), \nabla H_2(0, 0)\rangle > 0$, $i = 1, 2$.

For the internal corner collision:

(A-2) $\langle F_i(0, 0), \nabla H_1(0, 0)\rangle \geq 0$ or $\langle F_i(0, 0), \nabla H_2(0, 0)\rangle \leq 0$, $i = 1, 2$.

where $\langle ., .\rangle$ is a usual scalar product on R^n.