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Hybrid iterative scheme by a relaxed extragradient method for solutions of equilibrium problems and a general system of variational inequalities with application to optimization[☆]

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ABSTRACT

In this paper, we introduce a new iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the solutions of the variational inequality problem for two inverse-strongly monotone mappings. We introduce a new viscosity relaxed extragradient approximation method which is based on the so-called relaxed extragradient method and the viscosity approximation method. We show that the sequence converges strongly to a common element of the above three sets under some parametric controlling conditions. Moreover, using the above theorem, we can apply to finding solutions of a general system of variational inequality and a zero of a maximal monotone operator in a real Hilbert space. The results of this paper extended, improved and connected with the results of Ceng et al., [L.-C. Ceng, C.-Y. Wang, J.-C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008), 375–390], Plubtieng and Punpaeng, [S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2) (2008) 548–558] Su et al., [Y. Su, et al., An iterative method of solution for equilibrium and optimization problems, Nonlinear Anal. 69 (8) (2008) 2709–2719], Li and Song [Liwei Li, W. Song, A hybrid of the extragradient method and proximal point algorithm for inverse strongly monotone operators and maximal monotone operators in Banach spaces, Nonlinear Anal.: Hybrid Syst. 1 (3) (2007), 398-413] and many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping T of H into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$. A mapping f of C into H is called *contraction* if there exists a constant $k \in (0, 1)$ such that $\|fx - fy\| \le k\|x - y\|$, for all $x, y \in C$. Let ϕ be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $\phi : C \times C \longrightarrow \mathbf{R}$ is to find $x \in C$ such that

$$\phi(x, y) \ge 0 \quad \text{for all } y \in C. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $EP(\phi)$. Given a mapping $T : C \longrightarrow H$, let $\phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(\phi)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.1). In 1997 Combettes and Hirstoaga [1] introduced an iterative scheme of finding the best approximation to initial data when $EP(\phi)$ is nonempty and proved a strong convergence theorem.

The above formulation (1.1) was shown in [2] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(\phi)$. In other words, the $EP(\phi)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. (see also [3]).

Let $A : C \longrightarrow H$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The variational inequality has been extensively studied in the literature. See, e.g. [2,4–7] and the references therein. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.3}$$

for all $u, v \in C$. A mapping A of C into H is called α*-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.4}$$

for all $u, v \in C$. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. We denote by $F(S)$ the set of fixed points of S. For finding an element of $F(S) \cap VI(A, C)$, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.5}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, α_n is a sequence in $(0, 1)$, and λ_n is a sequence in $(0, 2\alpha)$. Recently, Nadezhkina and Takahashi [9] and Zeng and Yao [7] proposed some new iterative schemes for finding elements in $F(S) \cap VI(A, C)$.

The algorithm suggested by Takahashi and Toyoda [8] is based on two well-known types of method, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [10] for more details.

In 1976, Korpelevich [11] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_C(x_n - \lambda_n A\bar{x}_n) \end{cases} \tag{1.6}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k})$, C is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C in to $\mathbb{R}^n$. He proved that if $VI(A, C)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.6), converge to the same point $z \in VI(A, C)$.

Motivated by the idea of Korpelevichs extragradient method Zeng and Yao [7] introduced a new extragradient method for finding an element of $F(S) \cap VI(A, C)$ and proved the following strong convergence theorem.

Theorem 1.1 ([7, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be monotone and a k-Lipschitz-continuous mapping of C into H. Let S be a nonexpansive mapping from C into itself such that $\mathrm{F}(S) \cap VI(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:

(i) $\lambda_n k \subset (0, 1 - \delta)$ *for some* $\delta \in (0, 1)$*;*
(ii) $\alpha_n \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$.

Then the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same point $P_{\mathrm{F}(S)\cap VI(A,C)}x_0$ proved that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.