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Semistability of switched dynamical systems, Part I: Linear system theory☆

Qing Hui [a], Wassim M. Haddad [b,*]

[a] Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, United States

[b] School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, United States

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ABSTRACT

This paper develops semistability and uniform semistability analysis results for switched linear systems. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system's initial conditions. Since solutions to switched systems are a function of the system's initial conditions as well as the switching signals, uniformity here refers to the convergence rate of the multiple solutions as the switching signal evolves over a given switching set. The main results of the paper involve sufficient conditions for semistability and uniform semistability using multiple Lyapunov functions and sufficient regularity assumptions on the class of switching signals considered.

1. Introduction

An essential feature of multiagent network systems is that these systems possess a continuum of equilibria [1,2]. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a non-isolated equilibrium cannot be asymptotically stable. Hence, asymptotic stability is not the appropriate notion of stability for systems having a continuum of equilibria. For such systems possessing a continuum of equilibria, *semistability* [3,4] is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in the neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. It is important to note that semistability is not equivalent to set stability of the equilibrium set. Indeed, it is possible for trajectories to approach the equilibrium set without any trajectory approaching any single equilibrium [4].

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switching of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to switched dynamical systems. In this case, the vector field defining the dynamical system is a discontinuous function of the state and/or time, and hence, system stability should involve analysis of semistability of switched systems having a continuum of equilibria.

Building on the results of [1,5], in this paper we develop semistability and uniform semistability analysis results for switched linear systems. Since solutions to switched systems are a function of both the system initial conditions and

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* Corresponding author.
E-mail address: wm.haddad@aerospace.gatech.edu (W.M. Haddad).

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the admissible switching signals, uniformity here refers to the convergence rate to a Lyapunov stable equilibrium as the switching signal ranges over a given switching set. The main results of the paper involve sufficient conditions for semistability and uniform semistability using multiple Lyapunov functions and sufficient regularity assumptions on the class of switching signals considered. Specifically, using multiple Lyapunov functions whose derivatives are negative semidefinite, semistability of the switched linear system is established. If, in addition, the admissible switching signals have infinitely many disjoint intervals of length bounded from below and above, uniform semistability can be concluded. Finally, we note that the results of the present paper can be viewed as an extension of asymptotic stability results for switched linear systems developed in [6,7,5].

Although the results of this paper are confined to linear systems, nonlinear semistability theory for switched dynamical systems is considered in [8].

2. Switched dynamical systems

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathrm{Re}\,\lambda$ denotes the real part of $\lambda \in \mathbb{C}$, $(\cdot)^{\mathrm{T}}$ denotes the transpose, and $(\cdot)^{\mathrm{D}}$ denotes the Drazin generalized inverse. For $A \in \mathbb{R}^{n\times m}$ we write $\mathrm{rank}\,A$ to denote the rank of A, $\mathcal{N}(A)$ to denote the null space of A, $\mathcal{R}(A)$ to denote the range space of A, and for $A \in \mathbb{R}^{n\times n}$ we write $\mathrm{spec}\,(A)$ to denote the spectrum of A. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\mathrm{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$.

In this paper, we consider *switched linear systems* $\mathcal{G}_\sigma$ given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in \mathcal{S},\ x(0) = x_0,\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$, $\sigma : [0, \infty) \to \mathcal{P}$ denotes a piecewise constant switching signal, and $\mathcal{S}$ denotes the set of switching signals. The switching signal σ effectively switches the right-hand side of (1) by selecting different vector fields from the parameterized family $\{A_p x : p \in \mathcal{P}\}$. The *switching times* of (1) refer to the time instants at which the switching signal σ is discontinuous. Our convention here is that $\sigma(\cdot)$ is left-continuous, that is, $\sigma(t^-) = \sigma(t)$, where $\sigma(t^-) \triangleq \lim_{h\to 0^+} \sigma(t + h)$. The pair $(x, \sigma) : [0, \infty) \times \mathcal{S} \to \mathbb{R}^n$ is a *solution* to the switched system (1) if $x(\cdot)$ is piecewise continuously differentiable and satisfies (1) for all $t \geq 0$. The set $\mathcal{S}_p[\tau, T]$, $\tau > 0$, $T \in [0, \infty]$, denotes the set of signals σ for which there is an infinite number of disjoint intervals of length no smaller than τ on which σ is constant, and consecutive intervals with this property are separated by no more than T [5] (including the initial time). Finally, a point $x_{\mathrm{e}} \in \mathbb{R}^n$ is an *equilibrium point* of (1) if and only if $A_{\sigma(t)}x_{\mathrm{e}} = 0$ for all $\sigma(t) \in \mathcal{S}$ and for all $t \geq 0$.

We assume that the following assumption holds for (1).

Assumption 1. $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) - \{0\} \neq \emptyset$.

Let $\mathcal{E} \triangleq \{x_{\mathrm{e}} \in \mathbb{R}^n : A_{\sigma(t)}x_{\mathrm{e}} = 0, \sigma(t) \in \mathcal{S}, t \geq 0\}$. Then $\mathcal{E} = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p)$ and $\mathcal{E}$ contains an element other than 0. It is important to note that our results also hold for the case where $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \{0\}$. However, due to space limitations, we do not consider this case in the paper.

Definition 2.1. (i) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *Lyapunov stable* if for every switching signal $\sigma \in \mathcal{S}$ and every $\varepsilon > 0$, there exists $\delta = \delta(\sigma, \varepsilon) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq 0$. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq 0$.

(ii) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *semistable* if for every switching signal $\sigma \in \mathcal{S}$, x_{e} is Lyapunov stable and there exists $\delta = \delta(\sigma) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ and $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly semistable* if x_{e} is uniformly Lyapunov stable and there exists $\delta > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ uniformly in σ and $z \in \mathcal{E}$ is a uniformly Lyapunov stable equilibrium point.

(iii) The switched system (1) is *semistable* if all the equilibrium points of (1) are semistable. The switched system (1) is *uniformly semistable* if all the equilibrium points of (1) are uniformly semistable.

Next, we present the notion of semiobservability which plays a critical role in semistability analysis of linear dynamical systems. For details, see [9].

Definition 2.2 (*[9]*). Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}\left(CA^{k-1}\right) = \mathcal{N}(A). \tag{2}$$