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Averaged equations in a Hele-Shaw cell: Hierarchy of models

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ABSTRACT

In contrast to the classical Darcy law, Navier–Stokes–Darcy model of a flow in a Hele-Shaw cell includes small inertial and viscous forces in the plane of a cell. The displacement of viscous fluid by a less viscous one from a Hele-Shaw cell is studied numerically using these models. A distinguishing feature is the special microgravity conditions of displacement.

Miscibility of fluids together with the high speed of displacement (Peclet number tends to infinity) provides the absence of surface tension on the one hand and minimal manifestation of molecular diffusion on the other. Simulations based on the Darcy law and on the Navier–Stokes–Darcy model have been compared. The Navier–Stokes–Darcy model clearly exhibits a better consistency with the experimental data.

1. Introduction

Hele-Shaw cells are used to simulate seepage flows in porous mediums. Inner microscopic structure of a real porous medium is hard for a mathematical description and visualization difficulties arise during experimental researches. On the contrary, Hele-Shaw flows are easily visualized and described by rather simple mathematical models; therefore, the cell is applied as the simplest two-dimensional model of a porous medium.

Classical cells simulate porous medium with homogenous permeability [1]. The modified ones, incorporated with regular or randomized obstacles, are used to investigate the influence of inhomogeneity of porous matrix [2]. Many experiments were performed under microgravity conditions to avoid gravity effect on the displacement front structure [3].

A generalized two-dimensional Navier–Stokes–Darcy model of a flow in a Hele-Shaw cell is derived by the averaging of the three-dimensional Navier–Stokes equations for incompressible viscous fluid over a thin direction perpendicular to the cell's plates [4]. Let us obtain it briefly.

Introduce the averaged parameters of the flow and the deviations from the averaged values:

$$\langle f\rangle(x, y, t) = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} f(x, y, z, t)dz,$$

$$f' = f - \langle f\rangle, \quad f = \{u, \ v, \ w, \ p\}.$$

Integration of the Navier–Stokes equations over the Z direction leads to the Reynolds type equations, containing both the averaged parameters and the deviations:

$$\rho\left(\frac{\partial\langle u\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle u\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle u\rangle}{\partial y}\right) + \frac{\partial\langle p\rangle}{\partial x} = \mu\left(\frac{\partial^2\langle u\rangle}{\partial x^2} + \frac{\partial^2\langle u\rangle}{\partial y^2}\right) + \\ + \frac{2}{\delta}\mu\left[\frac{\partial u'}{\partial z}\right]_{z=\delta/2} - \rho\left(\frac{\partial\langle u'^2\rangle}{\partial x} + \frac{\partial\langle u'v'\rangle}{\partial y} + \frac{\partial\langle u'w'\rangle}{\partial z}\right),$$

$$\rho\left(\frac{\partial\langle v\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle v\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle v\rangle}{\partial y}\right) + \frac{\partial\langle p\rangle}{\partial y} = \mu\left(\frac{\partial^2\langle v\rangle}{\partial x^2} + \frac{\partial^2\langle v\rangle}{\partial y^2}\right) + \\ + \frac{2}{\delta}\mu\left[\frac{\partial v'}{\partial z}\right]_{z=\delta/2} - \rho\left(\frac{\partial\langle u'v'\rangle}{\partial x} + \frac{\partial\langle v'^2\rangle}{\partial y} + \frac{\partial\langle v'w'\rangle}{\partial z}\right),$$

$$\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y} = 0,$$

where $u, \ v, \ w, \ p$ – are velocity components and pressure, $\mu, \ \rho$ – fluid viscosity and density, δ – a gap between the cell's plates.

The Reynolds type system is unclosed. The terms with deviations are determining by concretizing the velocity profile between the cell's plates. If the changes of the flow parameters along the X and Y directions do not influence the velocity profile over Z, the velocity distribution between the cell's plates corresponds to a laminar Poiseuille flow:

$$u = \frac{3}{2}\langle u\rangle\left(1 - \frac{4z^2}{\delta^2}\right), \quad v = \frac{3}{2}\langle v\rangle\left(1 - \frac{4z^2}{\delta^2}\right), \quad w = 0.$$

It is important to point out, that the averaging procedure is conducted under another questionable assumption: every vertical section of the cell is completely filled with a single fluid. But for

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the displacement process this assumption does not take place, because a thin film of the displaced fluid is always remained on the cell's plates.

In case of a constant thickness film, the averaged model will be still valid with the changing of fluid viscosities on their «effective» values [1]. But for the real displacement, a more complex, three-layer, stratified flow is generated in a thin space between the Hele-Shaw cell's plates, and much more complicated procedure of averaging is needed to derive a plane model of motion. An alternative averaged model with transversal stratification of a flow between the cell's plates accounted is proposed by the author in [5].

Finally, the closed, two-dimensional Navier–Stokes–Darcy model is obtained:

$$\underline{\rho\left(\frac{\partial\langle u\rangle}{\partial t}+\frac{6}{5}\langle u\rangle\frac{\partial\langle u\rangle}{\partial x}+\frac{6}{5}\langle v\rangle\frac{\partial\langle u\rangle}{\partial y}\right)}+\frac{\partial\langle p\rangle}{\partial x} = -\frac{12\mu}{\delta^2}\langle u\rangle+\underline{\underline{\mu\left(\frac{\partial^2\langle u\rangle}{\partial x^2}+\frac{\partial^2\langle u\rangle}{\partial y^2}\right)}}, \tag{1a}$$

$$\underline{\rho\left(\frac{\partial\langle v\rangle}{\partial t}+\frac{6}{5}\langle u\rangle\frac{\partial\langle v\rangle}{\partial x}+\frac{6}{5}\langle v\rangle\frac{\partial\langle v\rangle}{\partial y}\right)}+\frac{\partial\langle p\rangle}{\partial y} = -\frac{12\mu}{\delta^2}\langle v\rangle+\underline{\underline{\mu\left(\frac{\partial^2\langle v\rangle}{\partial x^2}+\frac{\partial^2\langle v\rangle}{\partial y^2}\right)}}, \tag{2a}$$

$$\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}=0, \tag{3}$$

In the absence of the inertial (underlined once) and the viscous differential terms (underlined twice), one of the most fruitful particular cases of this model – the classical Darcy law arises:

$$\frac{\partial\langle p\rangle}{\partial x}=-\frac{12\mu}{\delta^2}\langle u\rangle,\quad \frac{\partial\langle p\rangle}{\partial y}=-\frac{12\mu}{\delta^2}\langle v\rangle, \tag{1b,2b}$$

Together with the continuity Eqs. (3), (1b) and (2b) form the closed system. It describes two-dimensional, potential flow with a pressure representing a potential. The Darcy law did not lose its relevance and is still applied in most of the works concerning Hele-Shaw cell.

Neglecting only inertial terms, one can obtain the Brinkman model which was highly demanded recently [6,7]. Without viscous differential terms the Navier–Stokes–Darcy model (1a), (2a), and (3) reduces to the Euler–Darcy model, also mentioned in some works [8]. Thus, *the hierarchy of models* can be built: from the *Darcy law*, through the *Euler–Darcy* and the *Brinkman* to the *Navier–Stokes–Darcy* model.

In the present work the displacement of viscous fluid by a less viscous one from a Hele-Shaw cell (Fig. 1a) is studied numerically using the Darcy law and the Navier–Stokes–Darcy model. It is well known, that such process is unstable: the displacing fluid (less viscous) tends to burst through a layer of the displaced one (more viscous), forming channels called viscous fingers. Their characteristic size depends on the several physical factors: the surface tension [9], the molecular diffusion [10] or the small viscous forces in a plane of a cell [4].

A large number of manuscripts were dedicated to the numerical simulation of displacement from a Hele-Shaw cell. Part of the authors was concentrated on the surface tension influence for immiscible fluids [11,12], another part – on the molecular diffusion effect for miscible ones [13,14].

2. Microgravity experiments

In the present work an unstable displacement of miscible fluids is considered, when the surface tension forces do not influence the finger's form and the molecular diffusion have no time to prove itself out. Such situation occurs in the case of the high speed displacement of water–glycerin mixture by water (miscible fluids) from a Hele-Shaw cell under microgravity conditions [15,16].

The results of two experiments (a and b: the first, c and d: the second) are shown in the Fig. 2. It is seen that the phase interface, though it has a rather complex shape, could be clearly detected during the displacement. Viscous fingers have finite width, which is defined by the gap between the Hele-Shaw cell's plates: experiments (a and b) and (c and d) differ only by the size of the gap δ. Analyzing all of the experimental data [15,16], one can also conclude, that the dependence of viscosity ratio M is weak and the average speed does not influence the displacement process at all. Thus, the gap between the cell's plates is the crucial parameter defining the interface structure.

The Darcy law does not include physical factor responsible for the viscous fingers width under these conditions. Linear analysis shows fingers absolute instability [17], so their width in numerical simulations is determined by the step of the grid. Analogous analysis based on the Brinkman model, containing the small viscous forces exerting in a plane of a cell, leads to an explicit dependence of the fingers width from the gap between the cell's plates. The similarity criterion is the ratio of the cell's width to the gap [4].

Fluid inertia (a part of the Euler–Darcy model) destabilizes fingers lateral surface, promoting their further destruction and even viscous bubbles formation [8]. The Navier–Stokes–Darcy model (1a), (2a), and (3) includes all the above-mentioned effects. Comparing simulations based on the Darcy law with the Navier–Stokes–Darcy ones is of the greatest interest from the author's point.

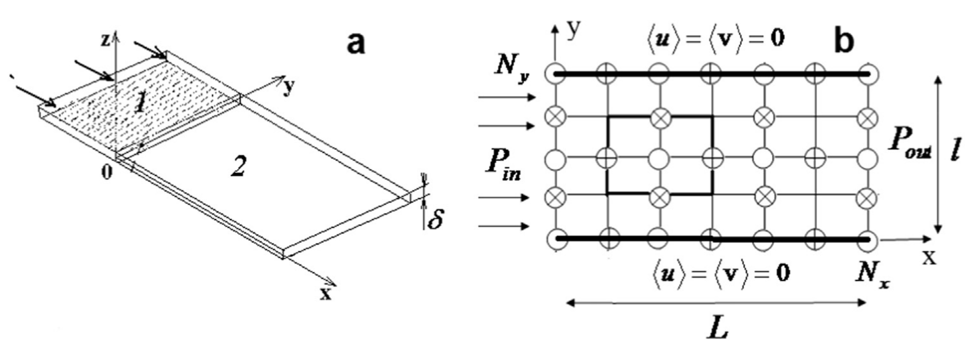

Fig. 1. To the displacement problem statement. (a): 3-D sketch of a Hele–Shaw cell, 1 – the displacing fluid, 2 – the displaced one. (b): 2-D domain Ω with the three kinds of nodes and boundary conditions.