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Transformed variables and hodographs in impulsive orbit transfer

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ABSTRACT

Recently a transformation of variables has been used for an object in a Newtonian gravitational field that linearizes the equations of motion. This transformation has been found useful for unconstrained orbital rendezvous and transfer problems.

This paper examines the geometry of these transformed variables for planar orbital transfer problems. The transformed initial, final, and transfer orbits are either points or circles with centers on a horizontal axis. Applied velocity impulses cause horizontal jumps between these points or centers and vertical jumps between points on the circular arcs. These transformed orbits are shown to have an equivalence to the well-known classical hodographs. Because of this equivalence the orbit equation can be represented by another set of linear equations in terms of the radial velocity, transverse velocity, and the reciprocal of the angular momentum.

1. Introduction

A recent approach to impulsive rendezvous and transfer problems that linearizes the equations of motion of a spacecraft between impulses has been applied to problems of optimal rendezvous [1] and orbit transfer [2] in a plane. Although this linearizing transformation is well known in the literature of the two-body problem, it has only recently been applied to orbit transfer or rendezvous. This is an example of rigorous linearization as discussed by Junkins and Singla [3] and extends and develops the $(1/r)$ transformation.

The first objective of this paper is to show that the linearizing transformation employed in this approach is not just a mathematical abstraction, but can be useful in establishing a new geometry of orbits and transfers that may be of use in visualizing orbital transfer and rendezvous. The second objective is to show that this geometry may provide an alternate way of calculating $\Delta\mathbf{v}$ requirements for certain transfers. This is demonstrated for some well-known transfers, the Hohmann transfer [4] and the bi-elliptical transfer [5–7]. The third objective is to show that the geometry of this transformation relates to two classical hodographs [8,9], one in a rotating coordinate frame and the other in an inertial coordinate frame. The former leads to another set of linear differential equations that describe the motion in terms of the velocity in the rotating frame and the reciprocal of the orbital angular momentum.

The organization of the paper is as follows. The following section presents the linearizing transformation. The next section presents the geometry of the boundary orbits and the transfer orbits. With this geometry, circular orbits become fixed points, elliptical orbits become circles, parabolic orbits become circles with one point missing on a vertical axis, and hyperbolic orbits are circular arcs on the right side of a vertical axis. Several examples are presented including the well-known Hohmann and bi-elliptic transfers. A general example is also presented that includes radial components in $\Delta\mathbf{v}$. It is demonstrated that the geometry of orbital transfers with transverse $\Delta\mathbf{v}$ differs from those having radial components. Finally we show how this geometry relates to the two classical hodographs and present another equivalent set of linear differential equations that describe the motion.

2. The impulse-free model

2.1. Changing the independent variable

The equation of motion of a spacecraft in an inverse-square gravitational field is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the position vector of a spacecraft measured from a center of attraction, $r = |\mathbf{r}| = (\mathbf{r}\cdot\mathbf{r})^{1/2}$ is its magnitude, μ is the product of the universal gravitational constant and a mass at the center of attraction, and the dot indicates differentiation with respect to time t. In polar coordinates (r, θ) this equation becomes

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$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{2}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}. \tag{3}$$

Multiplying Eq. (2) by r and integrating we obtain

$$r^2\dot{\theta} = h \tag{4}$$

where the constant h is the angular momentum of the spacecraft orbit. Using Eq. (4) to change the independent variable from t to θ and using a prime to denote differentiation with respect to θ the equations of motion can written as

$$r(\theta)r''(\theta) - 2r'(\theta)^2 = r(\theta)^2 - \mu r(\theta)^3/h^2 \tag{5}$$

$$h'(\theta) = 0. \tag{6}$$

These equations will be considered over a closed interval $\theta_0 \le \theta \le \theta_f$.

2.2. Linearizing the equations

We shall employ a change of variable $y(\theta) = \frac{1}{r(\theta)}$ used in recent work [1]. Based on this transformation we assign the new variables $y_1 = y$, $y_2 = -y'$, $y_3 = \frac{\mu}{h^2}$ and obtain the equivalent set of linear equations

$$y_1' = -y_2 \tag{7}$$

$$y_2' = y_1 - y_3 \tag{8}$$

$$y_3' = 0. \tag{9}$$

In previous work [1] we defined $y_1' = y_2$. We find that Eq. (7) used here is more convenient in the geometric work that follows in the next section. The linearity of these equations is a useful feature. They can be written as

$$\mathbf{y}' = A\mathbf{y} \tag{10}$$

where $\mathbf{y} = (y_1, y_2, y_3)^T$ and

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Replacing the fourth order system (2) and (3) by the third order system (7)–(9) loses information regarding the time t, however behavior with respect to time is not a part of the investigation of this paper.

2.3. State transition matrix

A fundamental matrix solution

$$\Phi(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 1-\cos\theta \\ \sin\theta & \cos\theta & -\sin\theta \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

associated with Eq. (10) is seen to satisfy

$$\Phi'(\theta) = A\Phi(\theta), \quad \Phi(0) = I \tag{13}$$

where I is the 3×3 identity matrix. For arbitrary $\theta_1, \theta_2 \in [\theta_0, \theta_f]$ this defines a state-transition matrix $\Phi(\theta_2 - \theta_1)$ satisfying

$$\mathbf{y}(\theta_2) = \Phi(\theta_2 - \theta_1)\mathbf{y}(\theta_1), \tag{14}$$

consequently an impulse-free trajectory is propagated by

$$\mathbf{y}(\theta) = \Phi(\theta - \theta_0)\mathbf{y}(\theta_0) \tag{15}$$

for any $\theta \in [\theta_0, \theta_f]$. Actually (13) and (14) are independent of the end values θ_0 and θ_f, thus (14) is valid for any real values of θ_1 and θ_2. We can therefore let $\mathbf{c}_0 = \Phi(-\theta_0)\mathbf{y}(\theta_0)$ and using properties of a fundamental matrix solution, an orbit through $\mathbf{y}(\theta_0)$ can be described as

$$\Phi(-\theta)\mathbf{y}(\theta) = \mathbf{c}_0 \tag{16}$$

for any real value of θ.

At any value of θ the velocity of the spacecraft is

$$\mathbf{v} = (v_r, v_\theta)^T = (\dot{r}, r\dot{\theta})^T. \tag{17}$$

In terms of the original variables the transformed variables are

$$y_1 = \frac{1}{r} \tag{18}$$

$$y_2 = \frac{\dot{r}}{h} \tag{19}$$

$$y_3 = \frac{\mu}{h^2} = \frac{\mu}{r^2 v_\theta^2}. \tag{20}$$

It can be observed that all three transformed variables are dimensionally the same. Values of the transformed variables at θ_0 and θ_f are obvious from those of the original variables at θ_0 and θ_f respectively.

3. Geometry of the transformed variables

Now we describe the geometry of transformed Keplerian orbits in the plane.

3.1. Circular arc (initial orbit)

We let $\bar{\theta} = \theta - \phi$ where ϕ is a constant. Using Eq. (16) we seek a transfer orbit $\Phi(-\bar{\theta})\mathbf{y}(\bar{\theta}) = \mathbf{c}_1$ that connects the initial orbit $\Phi(-\bar{\theta})\mathbf{y}(\bar{\theta}) = \mathbf{c}_0$ to the terminal orbit $\Phi(-\bar{\theta})\mathbf{y}(\bar{\theta}) = \mathbf{c}_f$.

Letting $\mathbf{c_0} = (c_{10}, c_{20}, c_{30})^T$ and replacing θ by $-\bar{\theta}$ in Eq. (12) yields the equations

$$\begin{aligned} &\cos\bar{\theta}y_1 + \sin\bar{\theta}y_2 + (1-\cos\bar{\theta})y_3 \\ &\quad = c_{10} - \sin\bar{\theta}y_1 + \cos\bar{\theta}y_2 + \sin\bar{\theta}y_3 = c_{20} \\ &y_3 = c_{30}. \end{aligned} \tag{21}$$

Since y_3 is constant we shall consider the transformed orbit as described by y_1 and y_2. Solving for y_1 and y_2,

$$\begin{aligned} y_1 &= (c_{10} - c_{30})\cos\bar{\theta} - c_{20}\sin\bar{\theta} + c_{30} \\ y_2 &= c_{20}\cos\bar{\theta} + (c_{10} - c_{30})\sin\bar{\theta}. \end{aligned} \tag{22}$$

Adding the squares of $y_1 - c_{30}$ and y_2, we obtain

$$(y_1 - c_{30})^2 + y_2^2 = d_0^2 \tag{23}$$

where

$$d_0 = [(c_{10} - c_{30})^2 + c_{20}^2]^{1/2}. \tag{24}$$

This is a circle having center $(c_{30}, 0)$ and radius d_0.

3.2. Circular arcs (terminal and transfer orbits)

Setting $\mathbf{c}_f = (c_{1f}, c_{2f}, c_{3f})^T$ the terminal orbit satisfies