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Comparison of mean and osculating stability in the vicinity of the (2:1) tesseral resonant surface

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ABSTRACT

We confront stability results over long time scales, considering alternately the averaged and the non-averaged theory to propagate the equations of motion of a celestial body orbiting the vicinity of the (2:1) tesseral resonant surface. This confrontation is performed using Fast Lyapunov Indicator stability maps. The benefit of such maps is threefold: (i) to reveal the whole phase space architecture and the consequences of the resonance overlap when several combinations of tesseral resonant parameters are accounted for, (ii) to perform a stability analysis on a whole phase space region, and (iii) to have a clear view of the possible impacts of the short-periodic effects removed during the averaging procedure. Our detailed numerical investigations conclude that the tesseral chaos is robust to the averaging procedure and the numerical methods used to propagate the equations of motion over such long time scales.

1. Introduction

The *averaging principle* or *averaging method* has seen a lot of improvements, mathematical justifications and rigorous developments [1] since its heuristic introduction by Lagrange in celestial mechanics and Van der Pol's works in mechanics. This perturbative method treats differential systems containing a small parameter which calibrates the perturbation's size of the original and non-perturbed system which is supposed to be integrable. Given a perturbed differential system, often written in the *standard perturbative form* [2] as

$$\begin{cases} \dot{x} = \epsilon X(x, y, \epsilon) \\ \dot{y} = \omega(x) + \epsilon Y(x, y, \epsilon), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ stands for the *slow variables* and $y \in \mathbb{T}^k$ the *fast variables*, $\epsilon \ll 1$, X and Y are 2π-periodic functions in y and are supposed to be analytic, the aim of the averaging method is to find new coordinates $(\overline{x}, \overline{y})$ in $\mathbb{R}^n \times \mathbb{T}^k$ such that the slow and the fast variables are separated. The solution reads as a power series of the small parameter ϵ:

$$\begin{cases} \dot{\overline{x}} = \epsilon A_1(\overline{x}) + \epsilon^2 A_2(\overline{x}) + \cdots \\ \dot{\overline{y}} = \omega(\overline{x}) + \epsilon B_1(\overline{x}) + \epsilon^2 B_2(\overline{x}) + \cdots \end{cases}.$$

When the previous calculus are performed at order 1 in ϵ, the term A_1 is the spatial average over the torus of the function X [2]. This method has been applied with success in spatial dynamics to define variables $(\overline{x}, \overline{y})$ free of short-periodic terms[1] but containing all the original long-periodic information. Due to the elimination of the short-periodic terms, the averaged equations of motion can been propagated numerically with a large step size (to the order of one day), several orders larger than those typically used when

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[1] Effects with a period on the order of the orbital period.

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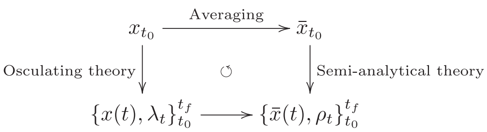
propagating the osculating motion, which is in practice very useful for long-term-analysis or long-term ephemeris.

If the first order averaged semi-analytical theory has already shown its capability in terms of orbit propagation accuracy when compared to the results of an osculating propagation [3], the compatibility problem between a stability analysis performed with the averaged or non-averaged equations has, to the best of our knowledge, never been addressed and is still an open problem, especially in the vicinity of resonant surfaces. This question is, from a dynamical and theoretical point of view, crucial and states the general problem of the short-periodic effects removed during the averaging procedure on long-term analysis. Consequently, we state here the question of the existence of the following, ideally commutative, diagram:



By x_{t_0} and $\bar{x}_{t_0}$ we denote the osculating and the corresponding mean initial state vectors at initial time t_0. These state vectors are propagated, following a numerical or semi-analytical approach, up to a final time t_f. By λ_t and ρ_t we refer to a numerical stability indicator associated with the proposed orbit. The main object of this paper is to study to what extent, or not, there is a compatibility and eventual link between the results of the stability analysis between $\{x,\lambda\}_t$ and $\{\bar{x},\rho\}_t$.

In this paper we examine the general problem applied to the (2:1) tesseral resonant motion, where GPS satellites are positioned. At this location, the orbital period is approximately equal to 12 h, half the rotational period of the Earth, leading to a resonant configuration.

The paper is organized as follows: In Section 2 we summarize the Hamiltonian part modeling the problem, and present the averaged Hamiltonian used for long-term motion study. The Hamiltonian is a 2 degree of freedom (hereafter noted DOF) Hamiltonian from which emerges chaos. The route to chaos is described by Chirikov's resonance overlap. In Section 3, we present the main results of the current work: the Fast Lyapunov Indicator (hereafter noted FLI) stability analysis obtained when propagating the averaged or non-averaged equations of motion. Several relevant stability maps are discussed.

2. Hamiltonian formulation of the problem

We recall in this section the general form of the Hamiltonian only when taking into account only the disturbing effect of the non-sphericity of the Earth. Since we are interested in the orbital evolution over long time spans, the Hamiltonian is averaged over fast variables. The averaged Hamiltonian is a 2-DOF Hamiltonian where Chirikov's resonance overlap occurs.

2.1. General formulation

We are dealing with the Hamiltonian representing the motion of a space object considering only the geopotential effect. Using the Delaunay's variables (L,G,H,l,g,h) related to the conventional Keplerian elements noted (a,e,i,Ω,ω,M), the Hamiltonian takes the form of the Keplerian Hamiltonian perturbed by the non-sphericity of the Earth:

$$\mathcal{H}=\mathcal{H}_{\text{Kep.}}+\mathcal{H}_{\text{Pert.}} \tag{2}$$

$$=-\frac{\mu}{2L^2}+\mathcal{H}_{\text{Pert.}}(L,G,H,l,g,h,\theta) \tag{3}$$

where the perturbing's part is given from Kaula's Earth development [4]

$$\mathcal{H}_{\text{Pert.}}(L,G,H,l,g,h,\theta)=\sum_{l\ge 2}\sum_{m=0}^{l}\sum_{p=0}^{l}\sum_{q=-\infty}^{+\infty}\Delta_{lmpq}\cos\left(\psi_{lmpq}\right) \tag{4}$$

where

$$\Delta_{lmpq}=\frac{\mu}{a}\left(\frac{r_E}{a}\right)^l F_{lmp}(i)G_{lpq}(e)J_{lm}, \tag{5}$$

$$\psi_{lmpq}=\left(l-2p+q-\frac{m}{s_0}\right)(M+\omega)+m(\lambda-\lambda_{lm})-q\omega, \tag{6}$$

$$\lambda=\frac{1}{s_0}(M+\omega)-(\theta-\Omega). \tag{7}$$

The $F_{lmp}(i)$-inclination and $G_{lpq}(e)$-eccentricity functions can be found in [4], r_E denotes the Earth's radius, μ the gravitational parameter and $J_{lm}=\sqrt{C_{lm}^2+S_{lm}^2}$ depends on the coefficients C_{lm} and S_{lm} describing the Earth's gravitational field. Indexes l and m are, respectively, the degree and order of the geopotential's development. Following the tradition [5,6], λ denotes the stroboscopic mean mode where s_0 is the closest integer of the ratio of the mean motion over Earth's rotational rate. The Hamiltonian given by Eq. (2) is non-autonomous due to the sidereal time θ. Finally, the quantity λ_{lm} is a phase variable depending only on the coefficient C_{lm} and S_{lm} whose definition can be found in [4].

Because the interest in this paper lies in the long-term analysis concerning the vicinity of the (2:1) tesseral resonant surface, the previous Hamiltonian, with various time scales, is now averaged over the fast variable M.

2.2. Averaging the Hamiltonian for the long-term motion

The perturbative part $\mathcal{H}_{\text{Pert.}}$ is split into the secular part $\mathcal{H}_{\text{sec.}}$ (terms independent of angles, those with $m=0$ and $l-2p+q=0$) and the resonant part $\mathcal{H}_{\text{res.}}$, containing terms dependent on θ. Averaging the Hamiltonian over the fast variable M is equivalent to retain in the resonant part only indexes (l,m,p,q) satisfying the $(\alpha:\beta)$ resonant condition with $(\alpha:\beta)=(2:1)$ in this work:

$$\frac{l-2p+q}{m}=\frac{\beta}{\alpha}=\frac{1}{2}. \tag{8}$$

The averaged Hamiltonian, that we continue to note $\mathcal{H}$, takes the form:

$$\mathcal{H}=\mathcal{H}_{\text{Kep.}}+\mathcal{H}_{\text{sec.}}+\mathcal{H}_{\text{res.}}, \tag{9}$$

and depends only of the angles λ and ω, a 2-DOF problem. When considered as isolated, i.e when only one resonant combination (l,m,p,q) is taken into account in Eq. (9), the dynamics may be reduced to a 1-DOF Hamiltonian by