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Solving the relative Lambert's problem and accounting for its singularities

Changxuan Wen*, Yushan Zhao, Baojun Li, Peng Shi

School of Astronautics, Beihang University, Xueyuan Road No. 37, Haidian District, Beijing, China

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ABSTRACT

A novel approach based on Lagrange's time equation and differential orbital elements is developed to solve the relative Lambert's problem for circular reference orbits. Compared with the conventional Clohessy–Wiltshire equation, the proposed method directly obtains differences of orbital elements between a transfer orbit and a reference orbit. This advantage enables us to account for singularities that occur in the relative Lambert's problem. The solved relative velocities depend on the five differential orbital elements. Accordingly, singularities can be attributed to any significant change in the semi-major axis, eccentricity, or orbital plane. Furthermore, appropriately adjusting initial and final relative positions eliminates some singularities. A numerical simulation based on the classic Lambert's formula for a rendezvous mission in closed range demonstrates the analytical results.

1. Introduction

The relative Lambert's problem (RLP) is an extensively considered issue in impulsive orbital maneuvers, such as rendezvous, interception, and formation initialization. Research on relative motion began in the early 1960s when Clohessy and Wiltshire obtained a linearized equation (CW equation) that models relative motion on a circular reference orbit [1]. They used a derivation similar to the one Hill adopted when studying the relative motion of the moon in the Earth–Sun system [2]. The CW equation was then adopted as an elegant model to solve the RLP [3], with most researchers focusing on the optimization problem of impulsive control for relative transfers. A commonly used optimization tool for impulsive rendezvous is the primer vector theory, which is developed by Lawden [4] and modified by Lion and Handelsman [5]. Prussing solved the minimum-fuel impulsive spacecraft trajectories for two-impulse rendezvous under a coast time long enough for one or more complete revolutions [6]. Impulsive control is also extensively used in close operations, such as formation establishment and reconfiguration and autonomous proximity [7,8].

These studies on minimizing characteristic velocity changes assume that mission objectives are achievable, but this assumption does not always hold for relative transfers. To illustrate, an inverse term on the transition matrix emerges in the solved relative velocity; this term may sharply increase the velocity when the matrix becomes singular. Xiang et al. first introduced the terms "unreachable point" and "high-propellant consumption area" to describe the conditions that surround a singular matrix and its neighborhood [9]. Zhu considered this problem the "singularity of RLP" [10]. Although both research groups obtained singularities by evaluating the determinant of the matrix, they failed to explain how and why this singularity occurs. To date, no adequate explanation of this problem has been offered. The CW approach is unsuitable for explaining this issue because it presents inadequate information on orbital elements.

Disregarding the CW approach as an option, we focus on the classic Lagrange's time equation (LTE) and the differential orbital element (DOE) model that describes

* Corresponding author.Tel.: +8613426296995.
E-mail address: wenchangxuan@gmail.com (C. Wen).

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relative motion. In the 1770s, Lagrange derived the analytical expression of the classic Lambert's problem for elliptical orbits, known as the LTE [11]. This equation also indicates the relations between the semi-major axis and two position vectors, provided that transfer time is given. Alfriend et al. introduced a geometry-based model, called the DOE model, to study relative orbital motion [12]. Given that this new model depicts the transformations between orbital elements and a relative state vector, it is often used to study the effects of perturbations and formation flying. Schaub and Gim adopted the DOE model to study relative motion under the effects of J_2 perturbation [13,14]. Jiang et al. investigated a relative boundary value problem by using this model to realize spacecraft formation flying [15].

The method for solving the RLP is conventional, but singularities of the RLP inevitably occur regardless of the approach used. Guibout and Scheeres used the generating functions and canonical transformations of Hamiltonian dynamics to solve the RLP [16,17]. Refs. [12] and [14] discuss state transition matrices in the form of DOEs for both circular and elliptical reference orbits. These matrices can also be used to solve the RLP and account for its singularities. However, the analytical derivation is somewhat complicated because the boundary conditions are the six independent equations of seven DOEs, and the transfer time is inexplicitly contained in the transition matrix. We instead propose a solution that combines the LTE and the DOE model, a method simpler than directly using the state transition matrix of DOEs. Orbital perturbations are excluded from our calculations because the duration of transfer is relatively short. The effects of perturbations are therefore negligible in final results.

Compared with the CW approach, the proposed solution enables us to determine all the DOEs between the transfer orbit and the reference orbit. Substituting the DOEs into the transformations between the DOEs and relative velocities yields the analytical results of the RLP. It shows that the solved relative velocities depend on only five DOEs. Consequently, singularity occurs when any one of these DOEs increases to infinity. Examining the analytical expressions of the DOEs enables us to readily obtain all singular conditions. This singularity results in high fuel consumption and failures in the relative navigation system, making such a system unacceptable for practical applications. However, appropriately adjusting the boundary conditions of the RLP can eliminate certain singularities.

The remainder of the paper is organized as follows. First, a brief review of the CW solution is presented to illustrate the problem discussed in this paper. We then derive the analytical expressions of all the DOEs in terms of two relative positions and transfer time, and obtain the initial and final relative velocities. On the basis of this new formulation, the singularities of the RLP are carefully analyzed through the evaluation of the singular conditions of each DOE. Finally, a numerical simulation is conducted to demonstrate the proposed solution and its singularity.

2. Review of the CW solution of the RLP

The RLP involves a transfer orbit and a reference orbit, as depicted in Fig. 1. We assume that the reference orbit is circular. In Fig. 1, $\mathbf{R}_i$ denotes the absolute position vectors under the Earth-centered inertial frame, and $\mathbf{r}_i$ represents the relative position vectors with respect to the reference orbit. Subscripts 1 and 2 correspond to initial and final times, respectively. $\Delta\theta$ is the transfer angle of the reference orbit during time interval $\Delta t = t_2 - t_1$. Given that Δt is proportional to the transfer angle for circular orbits, $\Delta\theta$ is adopted to represent the transfer time in the succeeding discussions. With these notations, the RLP can be described as the determination of initial and final relative velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ from the given $\mathbf{r}_1$, $\mathbf{r}_2$, and $\Delta\theta$.

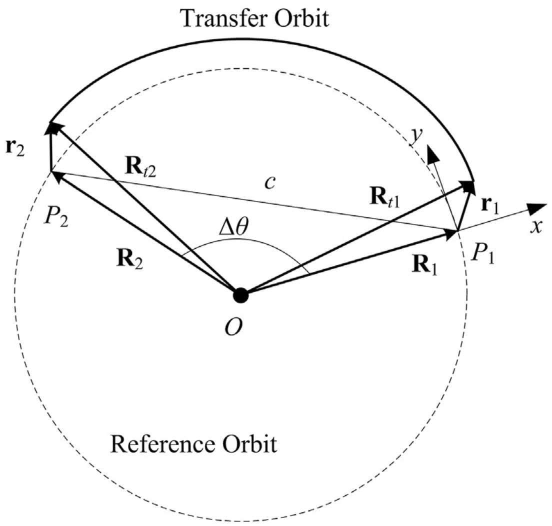

Fig. 1. Geometry of the RLP on a circular reference orbit.

The relative state vectors are written in terms of the components under the local-vertical–local-horizontal frame of the reference orbit

$$\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T, \quad \mathbf{v} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$$

The following equation is then readily obtained from the CW model,

$$\begin{bmatrix} \mathbf{r}_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}_{\Delta t} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{v}_1 \end{bmatrix} \tag{1}$$

where φ_{ij} denote the submatrices of the partitioned state transition matrix. Solving the equation yields the solution of the RLP as follows,

$$\begin{aligned} \mathbf{v}_1 &= \varphi_{12}^{-1}(\mathbf{r}_2 - \varphi_{11}\mathbf{r}_1) \\ \mathbf{v}_2 &= \varphi_{21}\mathbf{r}_1 + \varphi_{22}\varphi_{12}^{-1}(\mathbf{r}_2 - \varphi_{11}\mathbf{r}_1) \end{aligned} \tag{2}$$

The inverse matrix term φ_{12}^{-1} in Eq. (2) is undefined when the matrix is singular; this phenomenon is regarded as the singularity in the RLP. When singularity occurs, the relative velocity increases to infinity, except for some special cases. This issue is thoroughly explained and discussed later in this paper. Given that determinant of a singular matrix is zero, the singularity condition results in

$$\det(\varphi_{12}) = \frac{\sin\Delta\theta(8 - 8\cos\Delta\theta - 3\Delta\theta\sin\Delta\theta)}{n^3} = 0 \tag{3}$$

where n is the mean angular motion of the reference orbit.