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# Analytical theory for spacecraft motion about Mercury

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ABSTRACT

In the framework of the elliptic restricted three-body problem we develop an analytical theory for spacecraft motion close to Mercury. Besides the perturbations due to the gravity of the Sun and Mercury and the eccentricity of Mercury's orbit around the Sun, i.e., the elliptic restricted three-body problem, the theory includes the effects of the oblateness and the possible latitudinal asymmetry of Mercury, and is valid for any eccentricity of the spacecraft's orbit. The initial Hamiltonian defines a non-autonomous but periodic dynamical system of two degrees of freedom. The mean motion of the spacecraft and the time are averaged using two successive Lie–Deprit transformations. The resulting Hamiltonian defines a one degree of freedom system and depends upon three essential parameters. When the latitudinal asymmetry coefficient vanishes the flow of this system is entirely analyzed through the discussion of the occurrence of its (relative) equilibria and bifurcations in accordance with the parameters the problem depends upon. Frozen orbits of the initial system together with their stability are obtained related to the relative equilibria. If the latitudinal asymmetry of Mercury is taken into account, the equatorial symmetry of the problem is broken and introduces important changes in the dynamics. A variety of tests show a very good agreement between averaged and non-averaged models, and the reliability of the theory is further checked by performing long-term integrations in ephemeris.



## 1. Introduction

Analytical theories for mission designing of artificial satellites normally rest on simplified models that capture the majority of the dynamics. Thus, besides the Keplerian attraction, it is common to consider either the inhomogeneities of the potential of the central body, for instance for earth artificial satellites, or the third-body perturbation, e.g. for interplanetary missions, or a combination of both effects, as in the case of science missions about planetary satellites. The long-term dynamics reveals after averaging, and a full knowledge of the long-term dynamics is quite useful for mission-designing purposes.

In most cases it is enough to take the third-body perturbation in the Hill problem or the circular restricted three-body approximation—both models assuming the circularity of the primaries' orbit in its relative motion. However, the circular approximation does not fit to the case of Mercury, whose orbit around the Sun clearly exhibits a non-negligible eccentricity. Hence, the dynamics around Mercury must be studied in the context of the elliptic restricted three-body problem (ERTBP) [1] in spite of this model introduces a time dependence in the motion of the primaries. The ERTBP increases the number of degrees of freedom of the problem when compared with the circular approximation, and the extra degree of freedom must also be removed in the study of the long-term dynamics.

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The importance of including the eccentricity of the orbit of Mercury manifests in the first-order corrections related to the time averaging, as it has been recently shown [2]. However, to our knowledge, an analytical theory including the combined effects of the ERTBP and the non-spherical mass distribution of Mercury has not been considered yet. Because Mercury has a known oblateness with important effects on orbit stability and, probably, a latitudinal asymmetry without significant effects on stability but with effects on the shape of the science orbit that may be important, we feel compelled to develop a theory including both $J_2$ and $J_3$ harmonic coefficients of Mercury's gravitational potential, and the Sun gravitational perturbation in the ERTBP approximation. We call this variant of the ERTBP the zonal elliptic restricted three-body problem (ZERTBP).

The ZERTBP is Hamiltonian and, after formulating it as a perturbed two-body problem, the long-term behavior is studied by averaging. However, we depart from the classical approach of using the Lagrange planetary equations of an averaged perturbing function for describing the long-term dynamics. On the contrary, we use modern perturbation theory that besides the averaged equations also provides the transformation equations between averaged and non-averaged problems—equations that are essential to recover the short- and long-period effects lost in the averaging.

The theory is stepwise constructed by using Lie transforms [3]. The subtleties in the procedure introduced by the time dependency of the Hamiltonian are easily avoided by moving first to a higher dimensional problem using the homogeneous formalism [4]. This formalism assigns to the time the character of a new canonical variable. Then, we perform a Delaunay normalization [5] to remove the short-period effects associated with the mean anomaly of the orbiter. The subsequent elimination of the time by means of Lie transforms produces also the removal of the argument of the node, at least up to the third order of the theory—as it happened to the cometary and lunar cases of the ERTBP [6,2]. The resulting system is integrable and is discussed in terms of the spacecraft's mean elements. Specifically, the flow of the eccentricity vector, which decouples from the other mean elements, shows the existence of a variety of frozen orbits that might be useful in mission design.

When applying the analytical theory to the Sun–Mercury system we find a very good agreement between averaged and non-averaged models. But to further test the usefulness of our results, for a selected set of orbits, we propagate the initial conditions computed from the analytical theory in full ephemeris, using JPL files. The long-term propagations show that the analytical theory provides an accurate approximation of the real dynamics about Mercury.

## 2. Dynamical model

With the Sun–Mercury system in our mind, we tackle the motion of a massless point $S$ (the spacecraft) around an axially symmetric rigid body $m$ (Mercury) and under the gravitational attraction of a distant massive point $M$ (the Sun). This model is an extension to the Lunar case of the ERTBP [7], and may be useful in analyzing the motion of a spacecraft around a planet or a planetary satellite.

The derivation of the model below follows closely the approach of [2], and we provide it for the sake of completeness.

### 2.1. Relative motion

Let $(O,X,Y,Z)$ be the inertial frame of Fig. 1: the $(X,Y)$ plane is defined by the motion of the primaries $m$ and $M$, of masses $m$ and $M$, respectively, and the $Z$ axis is in the direction of its angular momentum vector. The Newtonian motion of $S$ is given by

$$\frac{\mathrm{d}^2\boldsymbol{R}}{\mathrm{d}t^2} = \nabla_S V(\boldsymbol{r}) - \frac{\mathcal{G}M}{q^3}\boldsymbol{q}, \tag{1}$$

where $\boldsymbol{R} = \vec{OS}$, $\boldsymbol{r} = \vec{mS}$, $\boldsymbol{q} = \vec{MS}$, $q = \|\boldsymbol{q}\|$, $V$ is the gravitational potential of $m$ and $\mathcal{G}$ is the gravitational constant. The motion of $S$ relative to $m$ is

$$\frac{\mathrm{d}^2(\boldsymbol{r}-\kappa\boldsymbol{\rho})}{\mathrm{d}t^2} = \nabla_S V(\boldsymbol{r}) - \mathcal{G}M\frac{\boldsymbol{r}-\boldsymbol{\rho}}{\|\boldsymbol{r}-\boldsymbol{\rho}\|^3}, \tag{2}$$

where $\boldsymbol{\rho} = \vec{mM} = \boldsymbol{r} - \boldsymbol{q}$, and $\kappa$ is the ratio mass of $M$ to the total mass of the system $\kappa = M/(M+m)$.

If $m$ and $M$ are far enough away to neglect the effects of the gravitational harmonics of $m$ on the motion of $M$, we find the two-body problem

$$\frac{\mathrm{d}^2\boldsymbol{\rho}}{\mathrm{d}t^2} = -\frac{\mathcal{G}(m+M)}{\rho^3}\boldsymbol{\rho}, \quad \rho = \frac{\alpha(1-\varepsilon^2)}{1+\varepsilon\cos f_M}, \tag{3}$$

where $\rho = \|\boldsymbol{\rho}\|$, the constants $\alpha$ and $\varepsilon$ are the semimajor axis and eccentricity of the Keplerian orbit, and $f_M$ is the true anomaly of the relative motion of $M$ with respect to $m$. Therefore,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \nabla_S V(\boldsymbol{r}) - \mathcal{G}M\left(\frac{\boldsymbol{r}-\boldsymbol{\rho}}{\|\boldsymbol{r}-\boldsymbol{\rho}\|^3} + \frac{\boldsymbol{\rho}}{\rho^3}\right). \tag{4}$$

### 2.2. Hamiltonian formulation

Under this approximation, the motion of the spacecraft $S$ is described by the Hamiltonian

$$\mathcal{K} = \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) - \frac{\mathcal{G}m}{r} - \mathcal{P}(\boldsymbol{r},\boldsymbol{\rho}), \tag{5}$$

where the conjugate momentum $\boldsymbol{u}$ to $\boldsymbol{r}$ is velocity in the inertial frame $\boldsymbol{u} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, and the perturbing function

$$\mathcal{P} = R_m + R_M \tag{6}$$

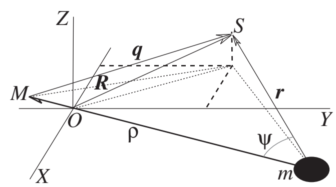


**Fig. 1.** Zonal elliptic restricted three-body problem.