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On vortex filament methods for linear instability analysis of aircraft wakes

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Linear stability analysis of vortical systems is discussed in a systematic manner using the classic inviscid vortex filament method. Well-known results are recovered as special cases of a unified linearization procedure. The symmetries of the vortex systems analyzed are exploited to obtain an analytical picture of different classes of growing and decaying eigenmodes. Finally, comparisons with viscous global linear theory reveal the limits of applicability of the vortex filament method for the instability analysis of realistic vortex systems.

1. Introduction

The importance of understanding instability of trailing vortex systems has been highlighted in a number of sources, e.g. [57], and derives from the desire to minimize commercial aircraft separation distances, especially during take-off and landing. Indeed, the vortical wake of a large aircraft may impose a hazard for any following aircraft of similar size or smaller. It is precisely the lack of understanding of the actual mechanisms of vortex decay, especially in the near mid-field, which leads to potentially conservative choices being made by the regulating authorities as regards aircraft separation distances. Although the peak of activity surrounding the introduction of the Boeing 747 and the Airbus A380, in the 1960s and 1990s of last century respectively, has now subsided, the ever-increasing aircraft density in a finite size of air space [13] keeps providing motivation for studies aiming at quantifying aircraft wake destruction mechanisms.

In order to address aircraft wake instability, a particularly suitable analysis methodology is that based on modeling vortices as a set of vortex filaments. This inviscid modal linear stability theory was introduced by von Kármán [64,65] and has been extremely useful in providing qualitative and on occasion quantitative information on linear flow dynamics of vortical systems. The vortex filaments theory has two variants: in its simplest form, introduced in the celebrated works of von Kármán [64,65] and also denoted as *point vortex stability analysis*, the two-dimensional limit is considered. A vortical flowfield is reconstructed by a number of straight and parallel vortex filaments which can be viewed as (two-dimensional) point vortices in equilibrium, whose stability is analyzed. In an aeronautical context, Donaldson and Bilanin [20] have used this methodology to analyze different configurations of four aligned point vortices and assess whether placement of vortices of a given circulation at specific distances results in a system of vortices in equilibrium.

The second of the two variants mentioned before, is concerned to the three-dimensional limit. In reality, in this limit, the previous set of points are a set of straight and parallel vortex filaments. If motion of the vortex filament in the third spatial direction is permitted, one may introduce a sinusoidal perturbation to the position of the vortex filaments and study whether deviations from this position are amplified or damped. The seminal work of Crow [19] on a counter-rotating vortex pair exemplifies this *vortex filament instability analysis*. In that work, the so-called long-wavelength instability (now commonly known as *Crow instability*) of the wake was first predicted; the favorable comparison with experiments and observations lent strength to the underlying theoretical concept. Similar analysis of Jiménez [31] for two vortices rotating in the same direction found this configuration to be stable, also in line with experimental observations. Many works followed, in which different initial vortex configurations are set, see e.g. Crouch [18], Rennich and Lele [53] and Fabre and Jacquin [21], among others.

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Beyond the simplified configurations of two or more aligned filaments exposed previously, a wide variety of other vortical configurations have been analyzed with respect to their instability using the same methodology, such as the arrays of vortices situated at the vertices of a regular polygon, a configuration referred to as *n-gon*, originally studied by Kelvin [33] and Thomson [62]. Havelock [28] presented a consistent analysis of the same configuration, while half a century later, Campbell [15] reviewed the topic and added more results. However, it was not up until recent times that the $N = 7$ configuration was shown to be unstable by Kurakin and Yudovich [40]. An analogous configuration, the instability of which was first analyzed by Joukowski [32] using the vortex filament method, is the vortex n-gon to which an additional vortex is added at its center and is taken to rotate in the opposite sense as compared with the sense of rotation of the vortices of the n-gon. This model is the cornerstone of analyses of instabilities in the downwash of a helicopter rotor, as well as those in the wake behind wind turbines. Additional interesting examples of application of point-vortex or vortex-filament instability analysis include the three vortices discussed by Aref [3,5], or other configurations involving a larger number of vortices [4]. Modifying the instability analysis methodology for straight vortex filaments, instability in a vortex helix can also be analyzed. This was first studied by Kelvin [34], and Levy and Forsdyke [44], while Ricca [54] and Kuibin and Okulov [39] added more light to the problem. Recently, instability in the wake of wind turbines was analyzed using an extension of the helical vortex filament method, in which multiple vortex helices with a common center are considered to model the tip vortices of the wind turbine blade, in the absence [49] or in the presence [50] of an additional counter-rotating vortex modeling the wake of the turbine nacelle.

While all the instability analysis methodologies briefly outlined above are inviscid in nature, viscous approaches are also numerous in the literature. Analysis of a single vortex by means of viscous local modal stability theory commenced by Lessen and Paillet [43] and continued in the works of Khorrami et al. [37], Khorrami [35,36], and Mayer and Powell [46]. More recently viscous global modal stability analysis of the linearized Navier–Stokes equations [24,60,61], has also been applied to the instability of vortex systems [12,25,26,29,30,51,56]. Direct Numerical Simulations (DNS) and Large Eddy Simulations (LES) have been used to perform viscous instability analysis of vortical flows: Abid and Brachet [1] analyzed a single vortex, Laporte and Corjon [41] studied a vortex pair, Rennich and Lele [53] a four vortex wake, Bristol et al. [11] investigated a vortex pair of unequal strength, and Nybelen and Paoli [48] studied co-rotating vortices. Others analyze instabilities solving initial value problems (IVP) as it is done by means of DNS or LES, but based on the linearized Navier–Stokes equations (LNSE), as in the work of Billant et al. [9] for a particular vortex pair. In addition, DNS of vortex particles offers exceptionally good performance for vortical flows, particularly for wakes [16,17,70] and for helical vortices [66]. Last but not least, experiments focusing on the stability of vortical wake flows have provided key information on novel instability mechanisms of vortex systems: the works of Leweke and Williamson [45], Laporte and Leweke [42] and the review of Williamson [69] for the cylinder wake offer examples of the impact that experimentation on the instability of vortex flow systems has had on the understanding of vortex dynamics and instabilities. It is worth mentioning that 3D-DNS, LES and the experiments, despite the quality of results that they produce, may require expensive equipment or large computational resources and, as such, can be performed only for a limited number of parameters. By contrast inviscid linear modal stability analysis by vortex filaments can be solved analytically in most of the cases or computed using negligible computing resources and, as such, it is well suited to perform parametric studies of the different values of the variables involved in the instability problem.

After this demonstration of the enormous amount of works related to vortex instabilities, it is important to highlight that the present work is focused on linear instability. The reader might rightly think that the non-linear analysis will be more accurate, but it is also true that for understanding the non-linear behavior is usually needed to understand first the linear one. In addition, the linear analysis provides in general more information about how to actuate on these instabilities to control the system, as well as non-linear analysis is usually orders of magnitude more expensive in computational terms. The authors have then focused on linear analysis, provided also that there is more than sufficient material to be shown already. The non-linear study deserves its own analysis, and it is left for future work of these or other authors.

The present contribution begins with a systematic review of the inviscid modal linear stability analysis by vortex methods, highlighting the most relevant studies in each area. Some of the results shown here are new and complete previous works, while some of the earlier results are reproduced in part here for consistency with the rest of the paper. The article is organized as follows. Section 2 discusses the theoretical concepts and deduces the equations solved, Section 3 is devoted to the corresponding results, and finally, the summary, conclusions and closing comments are offered in Section 4. Sections 2 and 3 are divided in three parts, one devoted to 2D inviscid analysis, another to its 3D counterpart and a third part is dedicated to the viscous theory and comparisons. The same scheme is followed both in the theoretical presentation and the results section. The two sections of inviscid theory (2D and 3D) begin with the theoretical concepts that lead to the general equations, which are subsequently made specific to a number of vortex configurations, starting from the most simple and advancing to the more complex.

2. Theory

The procedure carried out to perform the inviscid linear modal stability analysis by point vortices or vortex filaments is to obtain from the Biot–Savart law, the position derivatives (velocities) as function of the positions and intensities of all the vortices. These are then split into an equilibrium position and a small perturbation, and the equations are linearized. Additional assumptions can then be made, such as the selection of a homogeneous perturbation along the axial coordinate in the 3D case. The next step is to analyze the linear stability of the system by writing the time dependence as a complex exponential, and, obtain an eigenvalue problem for the complex frequencies. The values of these eigenvalues will determine the linear stability of the selected configuration of vortex filaments.

The inviscid vortex method for stability is based on idealized vortices that have an infinite value of vorticity concentrated at points or along filaments, and therefore, vortex size and vorticity distribution are not taken into account in this idealization. The importance of these effects is more relevant in the case of self-induction (the influence of one vortex over itself) and is negligible for the interactions among different filaments. This may seem to contradict the fact that the core size becomes more relevant when the vortices are closer, but the point is that in the equations (see Eqs. (17)), the vortex size only affects the self induction term, and the proximity effect is noticeable only because first the self induction acts, and then, the close vortex amplifies the effect. Actually, that is true in the 3D case, while in the 2D case, there is no mention at all on vortex size, the point vortices are infinitely small, and the results are completely independent of vortex size. Therefore, only the 3D case will be affected by these features, as self-induction does not exist in the 2D case.