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ORIGINAL ARTICLE

General exact solution of the fin problem with variable thermal conductivity

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Abstract In this paper, Lie point symmetry method is used to obtain the general exact solution of the second order nonlinear ordinary differential equation which governing heat transfer in rectangular fin with variable thermal conductivity. Some new forms of thermal conductivity are introduced and the associated exact solution is obtained in each case. The general relation among the fin efficiency, thermal conductivity and thermo-geometric parameter is obtained.

1. Introduction

A fin is a surface that extends from an object to increase the rate of heat transfer to or from the environment by increasing convection. Fins are most commonly used in heat exchanging devices such as radiators in cars, and heat exchangers in power plants. They are also used in newer technology such as hydrogen fuel cells.

The fin equation which is controlled the heat transfer of finis given by [1–19]

$$A_c \frac{d}{d\mathcal{X}}\left(K(T)\frac{dT}{d\mathcal{X}}\right) - Ph(T)(T-T_a) = 0, \tag{1}$$

where A_c is the cross-sectional area of the fin, $\mathcal{X}$ is the axial distance measured from the fin tip, $k(T)$ is the thermal conductivity of the fin, T is the fin temperature, P is the fin perimeter, $h(T)$ is the heat transfer coefficient and T_a is the ambient temperature.

Eq. (1) is obtained through some assumptions such as steady state operation with no heat generation, the fin tip is insulated, and one dimensional heat transfer.

The heat transfer coefficient $h(T)$ is given by [20]

$$h(T) = h_b\left(\frac{T-T_a}{T_b-T_a}\right)^n,$$

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where h_b is the heat transfer coefficient at the base of temperature, T_b is the temperature of the heat source which relate fin, the constant n indicates the mode of heat transfer.

After taking the previous assumption into account, Eq. (1) becomes

$$A_c \frac{d}{d\mathcal{X}}\left(K(T)\frac{dT}{d\mathcal{X}}\right) - Ph_b\frac{(T-T_a)^{n+1}}{(T_b-T_a)^n} = 0. \tag{2}$$

Eq. (2) can be made non dimensional by the set of the transformations [1–19]

$$\theta = \frac{T-T_a}{T_b-T_a}, x = \frac{\mathcal{X}}{L}, M^2 = \frac{Ph_bL^2}{K_aA_c}, k = \frac{K}{K_a}, \tag{3}$$

where L is a fin length and K_a is the thermal conductivity of the fin at the ambient temperature T_a.

Substituting Eq. (3) into Eq. (2), we obtain [17]

$$\frac{d}{dx}\left(k(\theta)\frac{d\theta}{dx}\right) - M^2\theta^{n+1} = 0. \tag{4}$$

The boundary conditions are given by [1–19]

1. At the fin tip ($\mathcal{X} = 0$), because of the fin is insulated, the change of temperature is

   $$\frac{dT}{d\mathcal{X}} = 0, \tag{5}$$

   From Eq. (3), Eq. (5) becomes

   $$\frac{d\theta}{dx} = 0 \text{ or } \dot{\theta}(0) = 0.$$

2. At the fin base ($\mathcal{X} = b = L$), fin temperature is the same temperature as the heat source T_b

   $$T(b) = T_b, \tag{6}$$

   From Eq. (3), Eq. (6) becomes

   $$\theta(1) = 1.$$

Here, the boundary conditions are

$$\theta(1) = 1, \qquad \dot{\theta}(0) = 0, \tag{7}$$

where, $\dot{} = \frac{d}{dx}$

Eq. (4) has been solved in many papers with different forms of the thermal conductivity $k(\theta)$ using approximation methods.

When $k(\theta) = 1$, series solutions of Eq. (4) are investigated using homotopy asymptotic method in Ref. [1] and by Adomian decomposition method in Ref. [2]. Exact analytical solution of Eq. (4) is obtained in Ref. [3] when $k(\theta) = 1$.

When $k(\theta) = (1 + \varepsilon\theta)$, Approximate solutions of Eq. (4) are investigated using homotopy analysis method in Ref. [4], by asymptotic analysis method in Ref. [5], by decomposition and evolutionary methods in Ref. [6] and for a special case when $n = 0$, series solutions of Eq. (4) are investigated using Adomian decomposition method in Refs. [7,8], by homotopy analysis method in Refs. [9–11], by homotopy perturbation method in Ref. [12], by modified decomposition method in Ref. [13], by variational iteration method in Ref. [14] and by differential transformation method in Refs. [15,21].

When $k(\theta) = \theta^\beta$, Approximate solutions of Eq. (4) are investigated using differential transform method in Ref. [16]. Exact analytical solution of Eq. (4) when $\beta = n$ is obtained in Ref. [17] but in the case $\beta \neq n$ the general solution of Eq. (4) was not obtained. In the case of steady state, exact analytical solution of Eq. (4) when $n = \beta = -\frac{4}{3}$ is obtained in Ref. [18].

In this paper, we will obtain the general exact solution of Eq. (4) for general function of thermal conductivity $k(\theta)$ using Lie point symmetry and the general relation of fin efficiency with the thermal conductivity and the thermo-geometric parameter M.

The following sections will be organized as follows: in Section 2, Lie point symmetry method will be used to obtain the general solution of Eq. (4). Also, the general relation among the temperature at the fin tip, the temperature gradient at the fin base, the mode of heat transfer n, the fin parameter M and the thermal conductivity at the fin base will be obtained. In Section 3, new cases of thermal conductivity are shown and the associated exact solution is obtained in each case. Also, the relation between the temperature gradient at fin base and the temperature at fin tip at these cases of thermal conductivity is discussed. In Section 4, we will study the fin efficiency.

2. Lie point symmetry method

It is known that the autonomous ODE Eq. (4) admits the Lie point symmetry generator [22,23]

$$X = \frac{\partial}{\partial x}. \tag{8}$$

The canonical coordinates $(r, v(r))$ associated with Eq. (8) are given by

$$r = \theta, \qquad v = x, \tag{9}$$

which prolong to

$$\frac{d\theta}{dx} = \frac{1}{v'}, \quad \frac{d^2\theta}{dx^2} = -\frac{v''}{(v')^3}. \tag{10}$$

Hence, Eq. (4) reduces to

$$-M^2\theta^{1+n} + k'(\theta)\left(\frac{1}{v'}\right)^2 - k(\theta)\frac{v''}{v'^3} = 0. \tag{11}$$

Let,

$$v' = \frac{1}{u}. \tag{12}$$

Substituting Eq. (12) into Eq. (11), we obtain

$$-M^2\theta^{1+n} + k'(\theta)u^2 + k(\theta)uu' = 0, \tag{13}$$

where, $' = \frac{d}{d\theta}$.