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Lagrangian approach to interfacial water waves with free surface

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ABSTRACT

This is a theoretical study on the interfacial water waves with a free surface in a two-layer system, where the lower fluid is taken to be heavier than the upper one. Lagrangian matching conditions are introduced for the physical fields separated by the interface. And a perturbation analysis is carried out to the third order to find the particle trajectory in the Lagrangian description. Observing the derived solution, a symbolic computation is introduced for obtaining the fifth order solution. The Lagrangian drifts, wave frequency, and set-up are also given as part of the solutions. Discontinuities across the interface are found for all of these physical quantities. A generalized set-up effect is found that the Lagrangian mean levels come near to both of the free surface and internal interface. Through some numerical calculations, it is shown that the larger density difference or relative wave height results in the larger drift velocity. The direction of particle motion in the upper layer is anti-clockwise in contrary to that in the lower layer. Better convergence for the Lagrangian solution than the Eulerian one is numerically demonstrated for the barotropic mode that the wave motion is dominated on the free surface.

1. Introduction

Interfacial water waves are of considerable practical interest in the ocean acoustics [1], physical oceanography [2] and marine ecology [3]. Internal waves propagating in the interior of the ocean may seriously cause damage to ocean structures, such as oil platforms and rail or road tunnels lying on the seabed. Internal waves will also lead to the advection of water particles along with nutrients, larvae and sediments. Therefore, knowledge of the particle dynamics in the internal waves in addition to the wave and current effects at the ocean surface is important.

Internal waves have been an active research field for decades. Studies have been performed theoretically, numerically, as well as experimentally. The study of internal waves propagating along a boundary between two layers of infinite thickness was pioneered by Stokes [4]. Hunt [5] obtained third-order formulas for the wave profile and the phase velocity in the case of two semi-infinite layers. Thorpe [6] extended Hunt's analysis to include the effects of finite fluid depth. Tsuji & Nagata [7] applied the Stokes' expansion technique to obtain the fifth-order solution for the internal waves moving between two layers of infinite thickness. Holyer [8] numerically calculated the maximum steepness for internal progressive waves. Funakoshi and Oikawa [9] studied two-dimensional long internal waves of large amplitude in a two-layer inviscid fluid between a rigid upper boundary and a rigid bottom. To describe internal waves for both the flat top-layer surface and the free-surface boundary conditions, Grimshaw and Pullin [10] utilized integral equation techniques to study the large-amplitude periodic waves of extreme form for internal waves. Umeyama [11] derived third-order asymptotic solution for the finite-amplitude interfacial wave and conducted experiments to validate the solution. Allalou et al. [12] studied the properties of three-dimensional periodic interfacial gravity waves. The above approximate analytic and numerical solutions to the internal wave problems were obtained using the Eulerian formulation.

Some analytical studies [4,13–20] have provided the particle trajectories for a wide variety of nonlinear waves in the Lagrangian description. These works have proved that there is no closed particle path throughout the fluid domain for the irrotational flows detailed therein. As a result, the mass transport can be well analyzed in the Lagrangian description for nonlinear progressive waves. Furthermore, the

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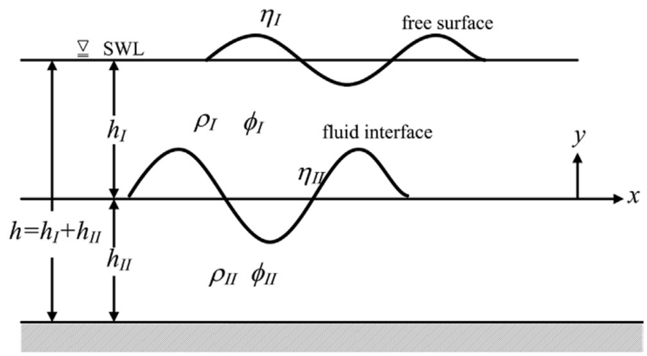

Fig. 1. Sketch of a two-layer fluid system.

particle motions predicted in these papers have been borne out by experimental evidence [21–23]. In addition, the third-order perturbation solutions [24–26] have been derived for nonlinear gravity-capillary and gravity waves with uniform or constant-vorticity current using the Lagrangian approach. To date, only a limited few theoretical solutions have been derived for internal waves in Lagrangian coordinates. Umeyama and Matsuki [27] gave a second-order solution of particle trajectory. However, they did not give the explicit expression nor include Lagrangian mean level and Lagrangian wave frequency. It has been shown that the Lagrangian approximations are more accurate than the Eulerian counterparts of the same order in the literatures [21,28,29]. However, reports on this improvement for the interfacial water waves in the Lagrangian description are rather limited.

In this paper, the Lagrangian formulations for waves problems [24–26] are further generalized to a two-layer fluid problem with a free surface. We aim to study particle trajectories of nonlinear interfacial waves completely based on the Lagrangian system. We look into the effect of the layers of different densities on an interfacial water wave, the motion of which is assumed to be inviscid, incompressible and irrotational. We will construct asymptotic expansions of the solution in powers of the relative wave amplitude. Approximate solutions are explicitly addressed up to the third order and symbolically computed up to the fifth order based on the technique recently introduced by Tsai, Chen and Hsu [30]. A detailed analysis of the particle trajectories for interfacial waves is then carried out. Also, the Lagrangian drifts, wave frequency, and set-up are all studied and explicitly given up to the third order.

The problem formulation, and the procedures for constructing asymptotic solutions are described in §2. In §3, we will derive equations for the properties of surface- and interfacial trajectories of particles up to third order. In §4, a symbolic computation is introduced for obtaining higher-order solutions. In §5 the surface profiles, Lagrangian drifts, wave frequency, mean level, and trajectories of particles in the two-layers fluid system are presented. Finally, some concluding remarks are given in §6.

2. Formulation of the problem

As sketched in Fig. 1, we consider a two-layer stratified system in which a lighter fluid layer rests on the top of a heavier fluid layer. The heavier fluid is bounded below by a rigid horizontal bottom. The lighter fluid is bounded above by a free surface. The depths of the two layers are h_I and h_{II}, respectively. Then, we can define the total depth as $h = h_I + h_{II}$. It is assumed that the fluids are homogeneous and separated by a sharp interface with little mixing. The densities of the two layers are ρ_I and ρ_{II}, respectively, and the density ratio is defined as $\rho_r = \rho_I/\rho_{II} < 1$. All physical quantities related to the upper fluid layer have the index *I*, while those related to the lower layer are indexed with *II*. We refer the equations of motion to Lagrangian coordinates (a, b), which are the undisturbed horizontal/vertical positions of a fluid particle. The y-axis is directed vertically upwards with the origin b=0 fixed at the equilibrium level of the interface. The current position of a particle, denoted by (x, y), is a function of a, b and time $t \geq 0$.

As in any two-layer system, two modes of wave motions (barotropic and baroclinic modes) are possible. The baroclinic mode, in which the interfacial wave amplitude is larger than that of the free-surface, is of interest and significance here. Following Fenton [31], the perturbation parameter is chosen as

$$\varepsilon = \frac{kH}{2}, \tag{2.1}$$

with H and $k = 2\pi/\lambda$ being the interfacial wave height and wavenumber. And λ is the wave length of the interfacial wave. Under the standard notation for the Jacobian, the mass conservation equation is given as

$$J = \frac{\partial(x_l, y_l)}{\partial(a, b)} = 1, \tag{2.2}$$

where the subindex l is used to distinguish between the upper and lower fluid domains as

$$l = \begin{cases} I & \text{forthefirstlayerin} h_I \geq b \geq 0, \\ II & \text{forthesecondlayerin} 0 \geq b \geq -h_{II}. \end{cases} \tag{2.3}$$

In practical derivation, it is convenient to take the time differentiation of Eq. (2.2) to have

$$\frac{\partial(x_{l,t}, y_l)}{\partial(a, b)} + \frac{\partial(x_l, y_{l,t})}{\partial(a, b)} = 0. \tag{2.4}$$