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Flexural gravity waves trapped in a two-layer fluid of finite depth

S. Saha, S.N. Bora*

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781039, India

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ABSTRACT

Trapped waves are of considerable interest in providing examples of discrete wave frequencies in the presence of a continuous spectrum. Under the usual assumptions of linear water wave theory, the existence of trapped modes supported by a submerged horizontal circular cylinder in a two-layer fluid of finite depth bounded above by a thin ice-cover and below by an impermeable horizontal bottom is investigated. The effect of surface tension at the surface of separation is neglected. In this case, two trapped waves are developed: waves with the higher wavenumber at the interface and waves with the lower wavenumber at the ice-cover. In this problem, a fifth-order boundary condition is satisfied at the upper surface which makes the problem more complex. Using multipole expansion method, an infinite system of homogenous linear equations is obtained. For a fixed geometrical configuration and a specific arrangement of a set of other parameters, the frequencies for which the value of the truncated determinant is zero are numerically computed and the trapped wavenumbers corresponding to those frequencies are obtained by using the dispersion relation.These trapped modes are compared with those for which the ice-cover gets replaced by a free surface. We also look into the effect of the variation of ice-parameters on the existence of trapped modes. Further, trapped modes in a homogenous fluid of finite depth bounded above by a thin ice-cover are recovered. Trapped modes due to a cylinder placed in either of the layers are mainly confined to the vicinity of interface and ice-cover only. These modes, in our case, exist with a cut-off value though there are trapped modes which are embedded in the continuous spectrum. So, above that cut-off value and far from interface and ice-cover, it is possible to have a unique solution to the radiation problem for the cylinder placed in either of the layers.

1. Introduction

Trapped modes are well-known in linear water wave theory. Most of the earlier works [1–4] suggest that these modes occur at discrete frequencies below a certain cut-off frequency and consist of local oscillations trapped near a long horizontal submerged body in finite or infinite water depth or over a sloping beach. The existence of trapped modes is closely related to the non-uniqueness of a forced-motion problem since the difference between two solutions of the problem of a long submerged cylinder making small forced harmonic oscillations at a trapped mode frequency is the trapped mode itself and the usual radiation condition is not sufficient to guarantee uniqueness.

It has been shown by John [5] that trapped modes do not occur near a partially immersed non-bulbous body in a fluid of uniform depth. Ursell [6] has shown that there are no two-dimensional trapped modes near a submerged circular cylinder in a canal. The existence of a trapped mode above a submerged horizontal circular cylinder in an infinitely deep water was first established by Ursell [1]. On the basis of full linearized theory of water waves, he showed that the existence of trapped waves depended upon the vanishing of a certain infinite determinant. The zeros of the determinant exist if the radius of the cylinder was small compared to the length of the waves. This is not a physical restriction, as has been shown by Jones [7], who proved that trapped waves exist for a number of geometries including a submerged cylinder of any radius and a rectangular shelf adjoining a region of greater depth. McIver and Evans [2] showed numerically that there always exists at least one trapped mode above a cylinder of arbitrary size and that further modes are possible as the cylinder approaches the free surface. This trapped wave was observed by them during an experimental testing based on the oscillations of a horizontal submerged cylinder. Evidence for the existence of trapped mode, where the governing equation is the Helmholtz equation, is provided by the works of Evans and Linton [4] and Callan et al. [3]. They computed the frequencies of trapped modes which occurred in the vicinity of a vertical cylinder extending throughout the water depth, placed on the centreplane of an open channel with the cylinder considered symmetric with respect to both centreplane and vertical plane perpendicular to it. The trapped mode solutions were antisymmetric about the centreplane. Subsequently Evans et al. [8] proved the existence of trapped modes for a general class of cylinders, placed symmetrically with respect to the channel centreplane with the motion antisymmetric about the plane. Majority of the works in trapped wave problems employed an inverse procedure to construct particular surface-piercing or submerged trapping structures in both

* Corresponding author. Tel.: +91 361 258 2604; fax: +91 361 258 2649.
E-mail addresses: swaroop@iitg.ernet.in, swaroopnandan@gmail.com (S.N. Bora).

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two and three dimensions [9–11]. These investigations show the existence of a new type of trapped modes which are embedded in a continuous spectrum.

In recent years the increase of human activities in the polar regions has amplified the necessity of investigations in the domain of ice cover dynamics. In order to understand the mechanism and the effects of wave propagation through the marginal ice zone in the polar regions, the ice-cover is modeled as a thin ice-sheet of which a very small part is immersed in water and is composed of materials having elastic properties. Unlike the case of plane gravity waves, in the presence of thin ice-cover, the upper surface boundary condition is of fifth order associated with the boundary value problem in which the governing equation is Laplace's equation which is not of the standard Sturm–Liouville type. Fox and Squire [12] developed a precise linearized model for the reflection and transmission process due to oblique waves at the margin of a sheet of shore fast sea ice. Chung and Fox [13] considered the interaction between the propagating waves and a semi-infinite ice sheet. They focussed on the calculation of the reflection of incident waves. Evans and Porter [14] analysed the problem of scattering of obliquely incident waves by a narrow crack in an ice-sheet floating in water of finite depth. They also used Green's function approach to solve the same problem. Another important reason for investigation of water wave problems in which water is covered by a thin ice-sheet is due to their application in the construction of very large floating structures, like floating oil storage bases, floating runways, etc.

It is not that only the wave motion at a free surface is of considerable interest, but there is also some significant interest in the generation of gravity waves at the interface of two fluids of different densities in which the upper fluid is covered by a rigid lid while the lower fluid is bounded below by a rigid horizontal bottom. Mohapatra and Bora [15] considered a three-dimensional problem involving the interaction of waves with a sphere. Multipole expansion method of Thorne [16] was used to evaluate the coefficients related to both heave and sway motions. The current authors have already investigated [17] the existence of trapped waves above a submerged horizontal circular cylinder for such a case by using the multipole expansion method used by Ursell [1]. They numerically computed the trapped mode frequencies by finding the zeros of a suitably truncated determinant.

In the above two classes of problems, there exists only one propagating wave. On the other hand, in the case of gravity wave propagation in a two-layer fluid having a thin ice-cover and an interface, two progressive waves exist which are generated at the upper surface and the interface. For example, (i) Bhattacharjee and Sahoo [18] obtained Fourier type expansion formulas and the related orthogonal mode-coupling relations for flexural gravity wave problems in a two-layer fluid; (ii) Mohapatra and Bora [19], by using linear water wave theory, investigated the scattering of oblique incident waves by small bottom undulations in a two-layer fluid where the upper surface was a thin ice-cover. Modified Helmholtz equation was solved, and the reflection and transmission coefficients were evaluated; (iii) Mohapatra and Bora [20] solved the scattering problem for a submerged sphere placed in one of the layers of the two-layer fluid and also computed the exciting forces for both horizontal and vertical directions; (iv) Mondal and Sahoo [21] analysed the effect of compressive force on flexural gravity waves in two-layer fluids. Wave characteristics for surface and interfacial modes in the cases of deep and shallow water were studied and generalized expansion formulas, along with associated orthogonal mode-coupling relations, were derived for the velocity potentials to deal with wave structure interaction problems in three dimensions in both the cases of finite and infinite water depths in channels of finite and semi-infinite widths.

The flexural gravity wave propagation in a two-layer fluid has been investigated to a reasonable extent as substantiated by the above works. However, to the best of the authors' knowledge, no investigation of flexural trapped waves in a two-layer fluid has taken place till date. Not many investigations have been carried out even for trapped modes for a two-layer fluid with a free surface. For example, (i) Kuznetsov [22] studied trapped modes in a channel spanned by a submerged cylinder in the (infinite) lower layer of a two-layer fluid. Using a perturbation method, he showed that there are two sets of trapped mode frequencies provided the density difference between the two layers is small; (ii) Linton and Cadby [23] investigated trapped modes above a horizontal cylinder in a two-layer fluid consisting of a layer of finite depth on top of an infinitely deep layer of greater density; (iii) Nazarov and Videman [24] derived the general sufficient condition for wave trapping in a two-layer fluid in which it was shown in particular that a trapped mode always exists when the submerged body intersects neither the free surface nor the interface.

The objective of the present work is to investigate whether a submerged horizontal circular cylinder in a two-layer fluid of finite depth bounded above by a thin ice-cover and below by an impermeable horizontal bottom supports trapped mode. In this case, two trapped waves are developed: the waves with higher wavenumber at the interface and the waves with lower wavenumber at the ice-cover. The wavenumbers of these waves are large compared to the wavenumbers of the corresponding gravity waves. Furthermore, in these problems, a fifth-order boundary condition is satisfied at the upper surface which makes the problem more complex. In order to examine the existence of trapped modes, multipole expansion method of [1], along with the properties of infinite system of linear equations, is used. For a fixed geometrical configuration, we numerically estimate the values of those frequencies for which the trapped modes exist. The trapped mode wavenumbers are plotted against the density ratio for different depths of the upper layer, different depths of the lower layer, submergence depths and different sets of ice-parameters. The dispersion curves for different geometrical configurations with a fixed set of the other parameters are also

2. Mathematical formulation of the problem

As the first step towards the formulation of our problem, the incident potential of the progressive wave is to be obtained by applying the governing equation and the related boundary conditions. Under the usual assumptions of linear water wave theory, the problem is considered in three dimensional Cartesian coordinate system with (x, y) plane in the horizontal direction and z-axis in the vertically upward direction. A two-layer fluid of finite depth is considered in which the upper layer is covered by a thin uniform ice-sheet modelled as a thin elastic plate, while the lower layer is bounded by a rigid infinite horizontal bottom. The upper fluid layer of constant density ρ_I occupies the region $0 < z < d$, $-\infty < x < \infty$, $-\infty < y < \infty$ with $z = d$ as the mean position of the thin ice-cover. The lower fluid of constant density ρ_{II} occupies the region $-h < z < 0$, $-\infty < x < \infty$, $-\infty < y < \infty$ with $z = 0$ as the mean position of the interface and $z = -h$ as the bottom surface (Fig. 1). The effect due to surface tension at the interface between the two fluids is neglected. With the fluid assumed to be inviscid and incompressible, and the motion irrotational, the fluid motion is described by the two velocity potentials $\Phi_j(x, y, z, t), j = I, II$. Let $\eta\,(x, y, z, t)$ and $\zeta\,(x, y, z, t)$ be the small displacement at the upper surface and the interface, respectively.

The velocity potentials $\Phi_j(x, y, z, t)$ satisfy the partial differential equation

$$\frac{\partial^2 \Phi_j}{\partial x^2} + \frac{\partial^2 \Phi_j}{\partial y^2} + \frac{\partial^2 \Phi_j}{\partial z^2} = 0 \quad \text{in the appropiate fluid region.} \tag{1}$$

The linearized kinematic conditions at the mean free surface and the mean interface, respectively, are given by

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi_I}{\partial z} \qquad \text{on} \qquad z = d, \tag{2}$$