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Focusing of surface waves by variable bathymetry

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ABSTRACT

Scattering of a monochromatic train of surface gravity waves incident on a finite region of arbitrary three-dimensional smoothly varying bathymetry is considered in this paper. The full three-dimensional linear water wave theory is approximated by the depth-averaged modified mild-slope equations and a Greens function approach is used to derive domain a integral equation for the function relating to the unknown surface over the varying bed. A simple but robust and effective numerical scheme is described to approximate solutions to the integral equation. The method is applied to bathymetries which exhibit focusing in the high-frequency ray-theory limit and used to illustrate that focusing occurs at finite wavelengths where both refractive and diffractive effects are included. Specifically, examples of elliptical and bi-convex lens bathymetries are considered.

1. Introduction

The idea of focusing of surface waves by underwater lenses was first proposed by [17]. The basic concept is rather simple: oblique waves are refracted by changes in depth and so as a wave passes from a depth h_1 to a smaller depth h_2, say, the refractive index n determined by $n = k_2/k_1 > 1$ allows oblique waves to 'straighten out', where k_1 and k_2 are the wavenumbers for travelling waves determined by the linear dispersion relation $K \equiv \omega^2/g = k_i \tanh k_i h_i$, $i = 1, 2$ [17] and subsequent later work by these authors used this idea to consider the focusing of surface waves by lenses which comprised horizontal underwater plates forming a 'Fresnel lens' (the type used in lighthouses and overhead projectors for example) in plan form, although a conventional bi-convex lens would work equally well. Thus incoming waves passing across the lens are transformed into a circular wave which converges at the focal point of the lens (see, for example, [26] and references therein). Linear theory and, later, nonlinear theory which accounted for the large amplitudes that arise in the vicinty of the focal point, were used with success in predicting large amplification of waves at focal points and these methods compared favourably with experiments in [26]. The theory used in this early work assumed that the effect of the depth dependence was simply manifested in a change in wavenumber which resulted in a two-dimensional wave equation in which the depth dependence was removed. Later, a numerical method based on fully three-dimensional linear theory was used by [20] to explore focusing by Fresnel and bi-convex lens [10] have used a different mechanism for focusing surface waves. Using a large periodic array of vertical cylinders whose diameters are much smaller than the incident wavelength, they appealed to homogenisation theory to argue large arrays of cylinders alter the wavenumber to create refraction. Using over 600 cylinders arranged to form a bi-convex lens, they demonstrated using direct numerical methods that focusing did indeed occur as homogenisation theory predicted.

Ref. [13] used similar ideas to previous authors, again employing a submerged horizontal plate in the shape of a lens to refract waves. In plan form the lens had an elliptical-arc leading edge and a circular-arc trailing edge. Here, the authors were exploiting ray theoretical result that incoming parallel rays entering an elliptical domain with refractive index $n = 1/\epsilon$ where ϵ is the ellipticity are exactly focussed on the far focal point P of the ellipse. By placing the centre of curvature of the trailing edge circular-arc at P the incoming rays refracted by the leading elliptical edge will be focussed on P. Experiments performed by [13] showed that this idea worked as predicted.

In this work we also take advantage of the elliptical lens focusing used by [13] and consider focusing of waves by an elliptical sea mount. Specifically, we examine the refraction of waves in otherwise constant depth h_1 incident on an elliptical mound, with a plateau at depth $h_2 < h_1$. According to geometric ray theory high frequency surface waves will be refracted by an abrupt change in depth and focus above the far focal point of the elliptical plateau (see Section 2 for a description of this apparently little known result). Of course, the change in depth could be effected by having waves pass across a submerged elliptical plate. Such a problem was considered by [30] and though they do mention focusing of waves, it is evidently clear that they are unaware of the ray theory result of exact focusing.

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When waves pass across raised bathymetry, refraction often results in amplification of waves behind the bathymetry and many papers have investigated this phenomenon. Three heavily cited papers in this area are due to [11,6,29] presumably because these papers include experimental data. In [11,29] amplification of waves by paraboidal and ellipsoidal shoals on a flat bed are considered [6] used a rotated ellipsoidal protrusion sitting on a linear sloping bed profile and produced numerical results based on mild slope equations, a ray theory approach showing the formation of a caustic behind the protrusion and experimental results. In none of these pieces of work consider geometries which predict perfect focusing under ray theory.

In this paper, we consider smoothly varying bathymetries motivated by the presence of perfect focusing as predicted by ray theory. A domain integral equation approach is developed to solve the problem based on the modified mild-slope equations (see, for example, [7]) which represent the three-dimensional fluid motion by two-dimensional depth-averaged equations based on the assumption that the gradient of the bed is small compared to the non-dimensional wavelength. It is perhaps worthy of note that the same assumption is used in a short-wavelength geometric optics approach to predict refraction over varying bathymetry, where ray paths are orthogonal to the phase lines $S(x, y)$ equals a constant where S satisfies the eikonal equation $S_x^2 + S_y^2 = k^2(x, y)$ and $k \tanh(kh(x, y)) = \omega^2/g$. See, for example, the description in [18]. The modified mild-slope approximation can be extended (e.g., [24,2]) to larger bed gradients and made increasingly accurate by the introduction of more vertical modes in the depth averaging procedure.

There is a difference in how rays bend when confronted with an abrupt change in depth and a gradual change in depth, though the final ray directions are the same. Hence the perfect focusing result described earlier and outlined in Section 2 for the vertically sided elliptical sea mount is lost once the change is depth is smoothed out. This is not an issue that we are overly concerned with as ray theory is introduced mainly for the purpose of motivation. Indeed, as we are concerned with surface gravity waves, the wavelengths considered here will be large enough that the defocusing effects of the gradual change in depth is probably not as important as the finite wavelength effect. Moreover, the formulation we propose allows for diffractive as well as refractive effects. Thus, in order to maximise the focusing of wave energy, we require a minimal amount of diffraction from the submerged bathymetry and this provides a secondary reason for the use of a smoothly varying bed.

In this problem we therefore consider bathymetries which rises gradually and smoothly from the open depth h_1 onto plateau of depth h_2. We will consider plateau of both elliptical and bi-convex lens shapes to demonstrate focusing effects. In Section 3 we describe the implementation of the mild-slope approximation to the fully three-dimensional problem and the formulation of domain integral equations from the reduced two-dimensional mild-slope equations using a Greens function approach, similar to that used in [23]. Section 4 outlines a simple but effective numerical discretisation method used to approximate solutions to the integral equations based on rectangular and circular based discretisations of the horizontal projection of the undulating bed. There are some similarities between our approach in this paper and the dual reciprocity boundary element method of [31] although our method appears much more straightforward both to formulate and implement numerically.

Finally, in Section 5, we produce a selection of graphical demonstrations of focusing of surface waves, illustrating focusing close to predictions from ray theory as the wavelength is decreased. In doing so, we indicate that an elliptical lens provides better focusing than the bi-convex lenses used previously by authors examining wave focusing. In addition, we assess the convergence of the numerical method and compare our results with the experimental results of [11,29].

2. Motivation: geometric ray theory

2.1. Elliptical lens

The following description can be found in [19]. Consider an elliptical domain with refractive index $n > 1$ and major axis $2a$, minor axis $2b$. Then the eccentricity is defined as $\epsilon = \sqrt{1 - b^2/a^2}$ and the focal points P and P' lie at $\pm a\epsilon$ (see Fig. 1). According to ray theory, a ray parallel to the major axis is incident on the ellipse, and makes an angle θ_i with the normal NN' to the boundary at O. The ray proceeds from O at an angle θ_r with respect to NN' where Snell's relates θ_i to θ_r by $\sin\theta_i/\sin\theta_r = n$. The ray intercepts the major axis at P and P' is the point at which a ray from P is reflected at O by the boundary onto the axis. Then $\angle ONP = \pi - \theta_i$ and by the sine rule $OP = nNP$. Also $\angle NOP' = \theta_r$ whilst $\angle ONP' = \theta_i$ and now the sine rule gives us $OP' = nNP'$. Adding these two results together gives $POP' = nPNP'$ and if this is to be independent of the point O, then we have P and P' at the focal points when we get $2a = n2a\epsilon$. In other words, we require $n = 1/\epsilon$.

When considering water waves in the short wavelength limit, a wave approaching the point O at which the elliptical boundary representing a change in depth is locally straight. By insisting that there is no change in the component of the wavenumber parallel to this boundary, we arrive at the relation $k_1 \sin\theta_i = k_2 \sin\theta_r$ where k_1 and k_2 are wavenumbers of propagating waves in depths h_1 and h_2 outside and inside the elliptical boundary. Thus in order to focus waves we require the relation

$$n = \frac{k_2}{k_1} = \frac{1}{\epsilon}, \tag{2.1}$$

to be satisfied. Hence, given the frequency ω, h_1 and h_2, we may use (2.1) to determine ϵ for focusing under the ray-theory limit.

2.2. Convex lens

The focal length, f, of a bi-convex lens is determined by the lensmakers' equation (see, for example, [9, p. 248, eqn. 6.2]),

$$\frac{1}{f} = (n-1)\left(\frac{2}{R} - \frac{(n-1)d}{nR^2}\right), \tag{2.2}$$

where n is determined by (2.1), d is the thickness of the lens from front to back and R is the radius of curvature of the lens.

3. Diffraction of waves by arbitrary three-dimensional bathymetry

3.1. Specification of the problem

The problem is described using Cartesian coordinates with the x and y axes lying in the mean free surface and z directed vertically upwards. The bed elevation is then given by $z = -h(x, y)$ where $h(x, y)$ is a continuous function with continuous derivatives over the varying part of the bathymetry, an arbitrary finite simply connected domain $(x, y) \in D$, and is such that $h(x, y) = h_1$, a constant, when $(x, y) \notin D$. Thus we require $h = h_1$ on $(x, y) \in \partial D$, the boundary of D, but can allow ∇h to be discontinuous across ∂D.

Under the usual assumption of linearised water wave theory, there exists a velocity potential given by $\Re\{(-\mathrm{i}g/\omega)\Phi(x, y, z)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$ where a time-harmonic dependence of angular frequency ω has been imposed and g is gravitational acceleration. We seek the time-independent complex potential $\Phi(x, y, z)$ which satisfies

$$(\nabla^2 + \partial_{zz})\Phi = 0, \quad -h(x, y) \le z \le 0, -\infty < x, y < \infty, \tag{3.1}$$