Coastal Engineering 109 (2016) 128–133

Contents lists available at ScienceDirect

Coastal Engineering

journal homepage: www.elsevier.com/locate/coastaleng

Improved Korteweg & de Vries type equation with consistent shoaling characteristics

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ARTICLE INFO

Article history:
Received 23 June 2015
Received in revised form 4 October 2015
Accepted 23 October 2015
Available online 27 November 2015

Keywords:
Kortweg & de Vries equation
Nonlinearity
Dispersion
Shoaling
Varying water depths

ABSTRACT

An improved Korteweg & de Vries type equation for uneven water depths with consistent linear shoaling characteristics is derived. The improvement of the equation is with respect to linear dispersion characteristics while consistency in linear shoaling characteristics is achieved via an exact agreement of the shoaling rate of the wave equation with that obtained from the principle of energy flux concept. Improvements in both linear dispersion and linear shoaling properties are demonstrated analytically and numerically by simulating a challenging experimental test case of nonlinear wave propagation over a submerged bar.

1. Introduction

The historic observation of John Scott Russell on horseback of a solitary wave in 1834 and his subsequent experiments in 1845 spurred the works on mathematical description of solitary waves (Miles, 1980). Boussinesq (1872) was the first to develop a wave equation with solitary wave solution. Lord Rayleigh (1876) made quite a similar derivation to the same purpose but in closing acknowledged the priority of Boussinesq's work. Korteweg and de Vries (1895) presented what might be termed the one-directional form of Boussinesq's one-dimensional wave model and showed that the equation admitted not only of solitary waves but also of a new class of permanent periodic waves named "cnoidal" waves as solution.

For more than half a century, till the early 1960s, the subject of solitary waves was quite dormant. Then, especially with the advent of computers, the interest in the Korteweg & de Vries equation or the KdV equation began growing. Miles (1981) gives a very illustrative graph of the number of citations of Korteweg and de Vries (1895) by year. Meanwhile, two dimensional forms of Boussinesq equations for varying bottom topography were derived first by Mei and Le Méhauté (1966) using the bottom velocity as the dependent variable and shortly afterwards by Peregrine (1967) using the averaged velocity instead of the somewhat ambiguous bottom velocity. These derivations were important for practical applications in coastal regions. In particular, Peregrine's Boussinesq equations for varying bathymetry have become almost the standard Boussinesq model of the coastal engineering community. Beginning from the 1970s Abbott and co-workers have developed numerical schemes for solving one- and two-dimensional wave propagation problems via Boussinesq models (Abbott, 1974; Abbott et al., 1973, 1978, 1984).

Witting (1984) made an outstanding contribution by introducing an improvement to the dispersion characteristics of Boussinesq type equations by means of a *new velocity* variable. At the same time his numerical treatment of the one dimensional equations included all the nonlinear terms though he rightly pointed out that the equations could not be called fully-nonlinear as the series expansion in vertical coordinate necessarily contained only finite number of terms. That is to say, the limited order of dispersion terms consequently limits the order of nonlinearity. This important point seems to be overlooked in some subsequent publications which claim full nonlinearity in Boussinesq equations, which in essence is not possible.

Madsen et al. (1991) added second-order terms to Boussinesq equations to improve the dispersion characteristics. Later, Madsen and Sørensen (1992) extended the improved equations to varying bathymetry. In the same vein, Beji and Nadaoka (1996) introduced the concept of partial replacement rather than addition and claimed to derive a consistent model, basing their arguments on the constancy of energy flux.

Nwogu (1993) gave an alternative derivation again of the Boussinesq equations with better dispersion properties by expressing the equations in terms of a velocity at an arbitrary water depth. Although never noticed the approach of Nwogu was indeed equivalent to that of Witting: Nwogu used the *velocity at an arbitrary depth* while Witting used a

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http://dx.doi.org/10.1016/j.coastaleng.2015.10.009
0378-3839/

new velocity expressed in terms of unknown coefficients. Nwogu's derivation is weakly nonlinear and for 2-D whereas Witting's derivation is strongly nonlinear and for 1-D; otherwise the two derivations stem from the same argument, which is essentially to use a different horizontal velocity variable other than a conventional one such as the mean velocity or the surface velocity.

With regard to the KdV equation a notable contribution was made by Benjamin et al. (1972). In a very formal and thorough analysis they showed that the term with three spatial derivatives representing the dispersion might be replaced by a term comprising two spatial derivatives and a time derivative. While the new form of the equation has the same formal justification it has mathematical and computational advantages over the former. Remarkably, before the formal justification of Benjamin et al. (1972), Peregrine (1966) used exactly the same form of the KdV equation in his numerical calculations of an undular bore. In time, replacing the nonlinear or dispersion terms (i. e. the second-order terms) with their equivalents has become a usual practice; Mei (1989 p. 550) enumerates two different nonlinear and four different dispersive terms, which in turn generate eight different KdV type equations.

In this work first a KdV type equation with mixed dispersion terms is derived from the combined form of the improved Boussinesq equations given by Beji and Nadaoka (1996). The improved Boussinesq equations are based on the application of partial replacement technique to the classical equations of Peregrine (1967) for varying water depth. Following the derivation of a generalized KdV equation with mixed dispersion and linear shoaling terms the linear shoaling gradient of the equation is compared with that obtained from the energy flux concept. Such direct comparability clearly indicates an intricate and inseparable link between the dispersion and the shoaling terms. Accordingly then the form of the KdV equation corresponding to Peregrine's classical Boussinesq model is found to produce a shoaling gradient in complete agreement with the energy flux concept. Though based on the unimproved Boussinesq equations the new type KdV equation possesses mixed dispersion and shoaling terms with improved characteristics. A numerical example based on the simulation of an experiment of Beji and Battjes (1994) is given to demonstrate the improved dispersion and shoaling aspects of the new equation.

2. Combined form of improved Boussinesq equations

By introducing the partial replacement technique to Peregrine's (1967) Boussinesq model for varying water depths Beji and Nadaoka (1996) gave the following continuity and momentum equations:

$$\eta_t + \nabla\cdot[(h+\eta)\mathbf{u}] = 0 \tag{1}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla\eta = \frac{(1+\beta)}{2}h\nabla[\nabla\cdot(h\mathbf{u}_t)] + \frac{\beta}{2}gh\nabla[\nabla\cdot(h\nabla\eta)] - \frac{(1+\beta)}{6}h^2\nabla(\nabla\cdot\mathbf{u}_t) - \frac{\beta}{6}gh^2\nabla\left(\nabla^2\eta\right) \tag{2}$$

where $\mathbf{u}$ is the vertically averaged or mean horizontal velocity vector with components (u, v) and η is the free surface displacement as measured from the still water level. $h = h(x, y)$ is the spatially varying local water depth and g is the gravitational acceleration. ∇ stands for two-dimensional horizontal gradient operator with components $(\partial/\partial x, \partial/\partial y)$ while subscript t denotes partial differentiation with respect to time. β is a non-dimensional scalar determined according to Padé approximation of the linear theory dispersion relation so that the resulting equations have better dispersion characteristics. Degenerate cases $\beta = 0$ and $\beta = -1$ indicate respectively Peregrine's original momentum equation and a full replacement of the dispersion term in the momentum equation.

The above equations, though nonlinear, may be combined to result in a single equation in terms of the free surface displacement η by appropriate approximations, which are not repeated here for the sake of brevity. The resulting equation is

$$\eta_{tt} = gh\nabla^2\eta + \frac{(1+\beta)}{3}h^2\nabla^2\eta_{tt} - \frac{\beta}{3}gh^3\nabla^2\left(\nabla^2\eta\right) + \frac{3}{2}g\nabla^2(\eta^2) + g\nabla h\cdot\nabla\eta + (1+\beta)h\nabla h\cdot\nabla\eta_{tt} - 2\beta gh^2\nabla h\cdot\nabla\left(\nabla^2\eta\right) \tag{3}$$

in which only the terms containing the first spatial derivative of the depth are retained. Truncation of higher depth gradients implies the use of *mild-slope* approximation, which is maintained throughout the work wherever necessary. 1-D form of Eq. (3) reads

$$\eta_{tt} = gh\eta_{xx} + \frac{(1+\beta)}{3}h^2\eta_{xxtt} - \frac{\beta}{3}gh^3\eta_{xxxx} + \frac{3}{2}g(\eta^2)_{xx} + gh_x\eta_x + (1+\beta)hh_x\eta_{xtt} - 2\beta gh^2h_x\eta_{xxx}. \tag{4}$$

The degenerate case $\beta = -1$ for constant depth gives the original derivation of Boussinesq (1872):

$$\eta_{tt} = gh\eta_{xx} + \frac{1}{3}gh^3\eta_{xxxx} + \frac{3}{2}g(\eta^2)_{xx}. \tag{5}$$

It is worthwhile to point out that Boussinesq's entire work was based on Eq. (5) (his Eq. (26)) and that he never gave his model separately as continuity and momentum equations.

3. Improved KdV type equation for varying depth

An improved KdV-like equation for uneven bathymetry is now derived. Derivation is based on the combined 1-D Boussinesq model, Eq. (4). First, introduce a co-ordinate system moving in the positive $x-$ direction with the non-dispersive phase velocity $C = \sqrt{gh}$ so that the evolutions of the wave form in this moving system is slow, permitting to write the following new co-ordinates:

$$\sigma = x - Ct, \qquad \tau = \varepsilon t \tag{6}$$

where ε is a small parameter indicating the weak changes of the wave form in time in the moving co-ordinate system. Expressing the terms in Eq. (4) in the new co-ordinate system gives

$$\begin{aligned}&\eta_{xx} = \eta_{\sigma\sigma}, \qquad \eta_{xxx} = \eta_{\sigma\sigma\sigma}, \qquad \eta_{xxxx} = \eta_{\sigma\sigma\sigma\sigma}\\ &\eta_{tt} = C^2\eta_{\sigma\sigma} - 2\varepsilon C\eta_{\sigma\tau} + \varepsilon CC_\sigma\eta_\sigma, \quad \eta_{xtt} = C^2\eta_{\sigma\sigma\sigma} - 2\varepsilon C\eta_{\sigma\sigma\tau} + \varepsilon CC_\sigma\eta_{\sigma\sigma}\\ &\eta_{txtx} = C^2\eta_{\sigma\sigma\sigma\sigma} - 2\varepsilon C\eta_{\sigma\sigma\sigma\tau} + 4\varepsilon CC_\sigma\eta_{\sigma\sigma\sigma} - 3\varepsilon C_\sigma\eta_{\sigma\sigma\tau}\end{aligned} \tag{7}$$

where the terms containing the spatial derivative of C have also been labeled by ε to indicate they are an order higher, and the terms proportional to ε^2 are all neglected. The neglect of these terms probably causes the resulting KdV type equations to lose the energy conservation characteristics that the Boussinesq equations possess. While the Boussinesq model of Beji and Nadaoka (1996) is consistent for any β values, the KdV type model derived from it is consistent for only $\beta = 0$ as is shown in Section 5. Substituting the expressions of Eq. (7) into Eq. (4) and re-arranging results in

$$-\varepsilon 2C\eta_{\sigma\tau} - \frac{1}{3}C^2h^2\eta_{\sigma\sigma\sigma\sigma} + \varepsilon\frac{2(1+\beta)}{3}Ch^2\eta_{\sigma\sigma\sigma\tau} - \frac{3}{2}g(\eta^2)_{\sigma\sigma} + \varepsilon CC_\sigma\eta_\sigma - \varepsilon gh_\sigma\eta_\sigma - \varepsilon\frac{(5-\beta)}{3}C^2hh_\sigma\eta_{\sigma\sigma\sigma} + \varepsilon\frac{5(1+\beta)}{2}Chh_\sigma\eta_{\sigma\sigma\tau} = 0 \tag{8}$$

where the last four terms are the so-called linear shoaling terms. The first two of these four terms originate from the continuity equation and may be put into the same form, and are kinematic in essence; while the last two terms originate from the dispersive terms or the