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On the nonlinear behaviour of Boussinesq type models: Amplitude-velocity vs amplitude-flux forms

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ABSTRACT

In this paper we consider the modelling of nonlinear wave transformation by means of weakly nonlinear Boussinesq models. For a given couple *linear dispersion relation-linear shoaling parameter*, we show how to derive two systems of nonlinear PDEs differing in the form of the linear dispersive operators. In particular, within the same asymptotic accuracy, these operators can either be formulated by means of derivatives of the velocity, or in terms of derivatives of the flux. In the first case we speak of *amplitude-velocity form* of the model, in the second of *amplitude-flux form*. We show examples of these couples for several linear relations, including a new amplitude-flux variant of the model of Nwogu (J. Waterway, Port, Coast. Ocean Eng. 119, 1993). We then show, both analytically and by numerical nonlinear shoaling tests, that while for small amplitude waves the accuracy of the dispersion and shoaling relations is fundamental, when approaching breaking conditions it is only the amplitude-velocity or amplitude-flux form of the equations which determines the behaviour of the model, and in particular the shape and the height of the waves. In this regime we thus find only two types of behaviours, whatever the form of the linear dispersion relation and shoaling coefficient. This knowledge has tremendous importance when considering the use of these models in conjunction with some wave breaking detection and dissipation mechanism.

1. Introduction

The modelling of wave transformation in the near shore region requires a physically correct description of both dispersive and nonlinear effects. The use of asymptotic depth averaged Boussinesq Type (BT) models for this task is quite common (Brocchini, 2013). These models, however, have to be used with much care. Quite often, enhanced weakly-nonlinear variants of these models, such as those proposed in (Abbott et al., 1978; Beji and Nadaoka, 1996; Madsen and Sørensen, 1992; Nwogu, 1993; Peregrine, 1967), are used outside of their range of applicability, e.g. when reaching breaking conditions. Fully nonlinear models are known to do a much better job in this regime (Grilli et al., 1994). However, also these models fail to actually include energy dissipation effects associated with wave breaking. To take into account these effects, either ad-hoc viscosity terms are included (Elnaggar and Watanabe, 2000; Kennedy et al., 2000; Nwogu, 1996), eventually based on PDEs for the vorticity transport equation (Briganti et al., 2004), or a coupling with the Shallow Water equations is introduced (Brocchini, 2013; Kazolea et al., 2014; Tissier et al., 2012; Tonelli and Petti, 2011). Despite the fact that they are theoretically well adapted only for small amplitude waves, weakly nonlinear BT models with wave breaking corrections provide in practice accurate results, even though clearly outside of their domain of validity (Brocchini, 2013; Kazolea et al., 2014; Roeber and Cheung, 2012).

The key to this success is actually the use of a properly designed and calibrated wave breaking model, which includes a breaking detection criterion and a dissipation mechanism. In the literature many wave breaking criteria exist, often classified in phase-averaged, using wave characteristics representative of one full phase of the wave (as the average wave height, length or period), and phase resolving ones, using informations computed from local characteristics of the wave. We refer the interested reader to Okamoto et al. (Okamoto and Basco, 2006) for a general review, and to (Brocchini, 2013; Kazolea et al., 2014; Roeber and Cheung, 2012; Tonelli and Petti, 2011) and references therein for further details on particular breaking criteria. Here we limit ourselves to observe that, in general, these criteria are based on physical information related to the shape and speed of the waves (height, slope, curvature, etc.) close to the breaking point. The challenge for a correct capturing of these features is, thus, the understanding of the genuinely nonlinear physics underlying breaking, as well as the understanding of the non-linear shoaling properties of the dispersive wave propagation model. A correct modelling of genuinely nonlinear effects is thus a research topic of high priority (Brocchini, 2013). In particular, the wave shoaling when approaching the nonlinear regime has a fundamental importance in this respect. Note that while the linear properties of the

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models can be thoroughly studied analytically (Dingemans, 1997), in the nonlinear case some properties, such as e.g. the shoaling behaviour, must be studied numerically.

Dispersive wave propagation can be modelled by means of several types of weakly nonlinear BT models. These all provide different approximations of the nonlinear wave (or Euler) equations. The design properties of these models are often the linear dispersion relation and the shoaling coefficient. These are optimised to be as close as possible to those of the linear wave theory for the range of wave numbers relevant for the applications sought. Given a linear dispersion relation and linear shoaling coefficient, it is known that two nonlinear sets of Partial Differential Equations (PDEs) can be formulated, both degenerating to the same linearized system. Denoting by a the wave amplitude, h the mean water level, and λ the wavelength, these two models are alternate forms within the same asymptotics in terms of the nonlinearity $\varepsilon = a/h$ and dispersion $\sigma = h/\lambda$ parameters. The main difference lies in the nature of the higher order derivatives, which can either be applied to the velocity u, or to the flux $q = du$, d denoting the depth. These formulations are referred here to as *amplitude-velocity*, and *amplitude-volume flux* forms. Examples of such couples for some dispersion relations are given in (Dingemans, 1997).

To gain a better understanding in the properties of weakly nonlinear BT models, this paper presents a thorough analytical and numerical characterization of their nonlinear behaviour. For a given couple linear dispersion relation-linear shoaling parameter, we start by recalling how to construct, within the same asymptotic accuracy, two nonlinear sets of PDEs: the first one in amplitude-velocity form, the other one in amplitude-volume flux form. The theory is applied to four linear relations corresponding to the models of Peregrine (Peregrine, 1967) and to the enhanced models of Beji and Nadaoka (Beji and Nadaoka, 1996), Madsen and Sørensen (Madsen and Sørensen, 1992) and Nwogu (Nwogu, 1993). For each of these models, we give the corresponding alternate formulations. We obtain that the amplitude-volume flux form of the Peregrine system leads to the model used by Abbott in (Abbott et al., 1978), the amplitude-volume flux form of the Beji and Nadaoka model is equivalent to a modified form of the Madsen and Sørensen model, and vice versa that the amplitude-velocity form of the latter can be obtained by a small modification of the model of (Beji and Nadaoka, 1996). Finally, for the equations of Nwogu, we derive a new BT system which is the corresponding amplitude-volume flux form. We then study these models, and the main result of the paper can be summarized as follows: *while in the linearized case four types of behaviours are observed, corresponding to the given four couples of linear dispersion relation-shoaling coefficient, when approaching the nonlinear regime, only two type of behaviours are observed, which are independent on the linear dispersion relations and shoaling parameters, and depend only on whether the model is formulated in amplitude-velocity or amplitude-volume flux form*. This observation is confirmed by both theoretical arguments and numerical results.

The present study gives important insight in the behaviour of BT models, especially in view of the applications of breaking detection criteria. In particular, our result shows that these criteria must take into account not only the type of breaking expected in the flow but also the underlying form of the propagation model. For simplicity, we only consider here models well adapted to the near shore range (reduced wave numers $kh \leq \pi$), however very similar arguments can be used to study deep water variants (Madsen and Schaffer, 1998).

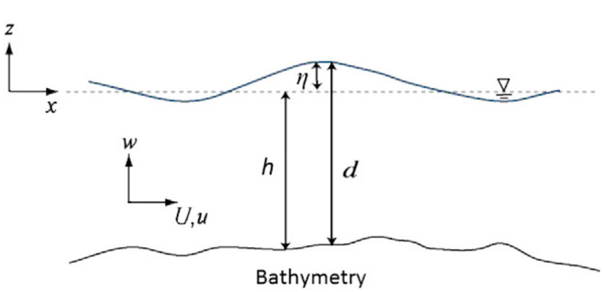

Fig. 1. Sketch of the free surface flow problem, main parameter description.

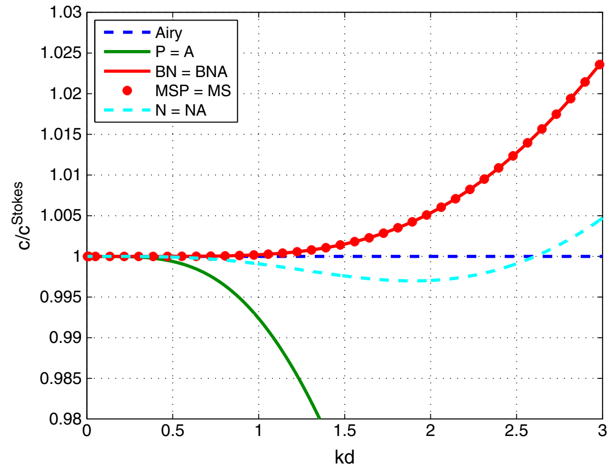

Fig. 2. Phase velocity ratio for all the models: P stands for Peregrine (Eq. (3)), A for Abbott (Eq. (8)), BN and BNA for Beji–Nadaoka and Beji–Nadaoka–Abbott (Eqs. (11)) and (12) resp.), MS and MSP for Madsen–Sørensen and Madsen–Sørensen–Peregrine (Eqs. (14) and (17) resp.), and N and NA for Nwogu and Nwogu-Abbott (Eqs. (22) and (26) resp).

The structure of the paper is the following. In Section §2. we present the derivation of weakly nonlinear Boussinesq equations, and we discuss the construction of models in amplitude-velocity, and amplitude-flux forms for different particular cases. Section §3 presents the theoretical analysis of the systems of PDEs obtained, and in particular the analysis of the propagation of higher order harmonics, which gives an indication of the non-linear behaviour of the models. Finally, numerical tests in both the linear and nonlinear regime are discussed in Section §4. The paper is ended by conclusive remarks and by an overview of future works.

2. Weakly nonlinear Boussinesq type models

We review a certain number of weakly nonlinear Boussinesq type (BT) models. We recall that these models are obtained as depth averaged asymptotic approximations of the incompressible Euler equations. In particular, if a denotes a reference wave amplitude, h_0 a reference

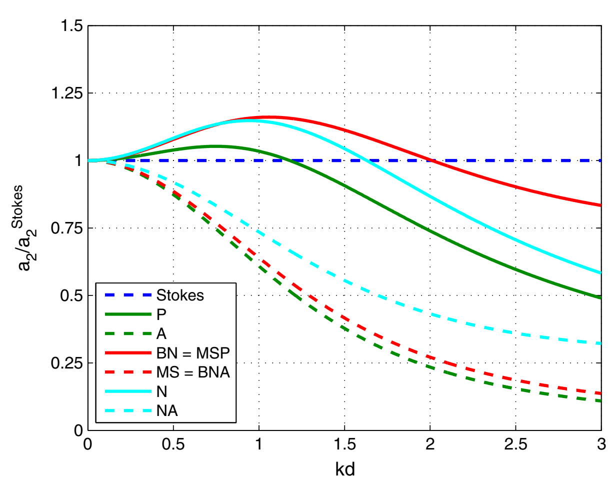

Fig. 3. Ratio of the second harmonic a_2/a_2^{Stokes} for the models considered. Continuous line: amplitude-velocity models. Dashed lines: amplitude-flux models. Refer to Fig. 2 for the legend.