Computers and Chemical Engineering 84 (2016) 568–598

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

New *a priori* and *a posteriori* probabilistic bounds for robust counterpart optimization: I. Unknown probability distributions

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A R T I C L E I N F O

Article history:
Received 10 July 2015
Received in revised form
11 September 2015
Accepted 11 September 2015
Available online 26 September 2015

Keywords:
Robust counterpart optimization
Optimization under uncertainty
Probabilistic bounds
Mathematical modeling

A B S T R A C T

Optimization problems often have a subset of parameters whose values are not known exactly or have yet to be realized. Nominal solutions to models under uncertainty can be infeasible or yield overly optimistic objective function values given the actual parameter realizations. Worst-case robust optimization guarantees feasibility but yields overly conservative objective function values. The use of probabilistic guarantees greatly improves the performance of robust counterpart optimization. We present new *a priori* and *a posteriori* probabilistic bounds which improve upon existing methods applied to models with uncertain parameters whose possible realizations are bounded and subject to unspecified probability distributions. We also provide new *a priori* and *a posteriori* bounds which, for the first time, permit robust counterpart optimization of models with parameters whose means are only known to lie within some range of values. The utility of the bounds is demonstrated through computational case studies involving a mixed-integer linear optimization problem and a linear multiperiod planning problem. These bounds reduce the conservatism, improve the performance, and augment the applicability of robust counterpart optimization.

1. Introduction

The parameters of applied mathematical models often have multiple possible values which they can achieve due to limited information or measurement error. The solutions and objective function values produced from optimizing models with uncertain parameters can vary greatly based on which values the uncertain parameters realize. One method of handling uncertain parameters is to guarantee feasibility of the constraints for all possible parameter points contained within deterministically defined parameter spaces, called uncertainty sets; the corresponding optimization model is known as a robust counterpart. For bounded uncertainty, that is, for the case where upper and lower bounds on each uncertain parameter are known, Soyster (1973) formulated the so-called “worst-case” robust counterpart, where each constraint’s uncertainty set is defined to include all possible parameter points; the probability that parameters would realize values rendering an optimal solution infeasible is zero. El Ghaoui and Lebret (1997) applied robust optimization methods to uncertain least-squares optimization problems, and El Ghaoui et al. (1998) solved uncertain semidefinite optimization problems.

Less conservative solutions can also be obtained for models with bounded or unbounded uncertain parameters, where the uncertainty sets can be defined with a corresponding probability of constraint violation greater than zero. For bounded uncertainty in linear optimization models (LPs), Ben-Tal and Nemirovski (2000) proposed the interval + ellipsoidal uncertainty set and derived a method to provide an upper bound on constraint violation given the size of the uncertainty set. Bertsimas and Sim (2004) proposed and characterized the interval + polyhedral uncertainty set, which yielded a linear robust counterpart. The robust counterpart optimization framework was extended to mixed-integer linear optimization models (MILPs) by Floudas and coworkers (Janak et al., 2007; Lin et al., 2004; Verderame and Floudas, 2009a,b) with parameters subject to uncertainty with known or unknown distributions, including unbounded probability distributions.

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http://dx.doi.org/10.1016/j.compchemeng.2015.09.014

In a traditional robust counterpart optimization framework, the quality of an optimal solution to the robust counterpart relies heavily on *a priori* methods that define uncertainty sets which are guaranteed to satisfy a particular upper bound on the probability of constraint violation; a tighter probabilistic bound would allow the imposition of a smaller uncertainty set with the same guarantee of feasibility and can drastically improve the objective function values. In other words, a tighter probabilistic upper bound on constraint violation leads to less conservative solutions. Both Ben-Tal and Nemirovski (2000) and Bertsimas and Sim (2004) derived *a priori* methods that provided an upper bound on the probability of constraint violation based on uncertainty set size for interval + ellipsoidal and interval + polyhedral sets, respectively. Kang et al. (2013) utilized distribution-dependent bounds with the interval + polyhedral uncertainty set. Li et al. (2011) extended these bounds to apply to other uncertainty sets so that a variety of *a priori* bounds were available for box, ellipsoidal, interval + ellipsoidal, polyhedral, and interval + polyhedral uncertainty sets. An alternative approach is to characterize the probability of constraint violation of a particular solution, that is, to obtain a probabilistic guarantee *a posteriori* (Kang et al., 2013; Li and Floudas, 2014; Li et al., 2012; Paschalidis et al., 2008), which would yield a tighter bound than an *a priori* bound of equivalent structure. An *a posteriori* bound can be incorporated into the robust counterpart, permitting feasibility only when the bound is met, as an alternative to an uncertainty set, though this yields a highly nonconvex optimization problem (Li and Floudas, 2014). Li and Floudas (2014) proposed an iterative method which yields better solutions than those obtained from the traditional robust counterpart optimization framework (i.e., defining uncertainty sets via *a priori* bounds), by converging them towards the tighter results given by *a posteriori* bounds. Utilizing tight *a priori* and *a posteriori* bounds with this iterative method can provide drastically improved solutions when compared to the worst-case approach, as well as the traditional one-pass robust counterpart optimization framework.

We present fundamental theoretical results on new *a priori* and *a posteriori* bounds on the probability of constraint violation which improve upon existing methods. Situations for which one or more of the new bounds are applicable include (i) bounded, symmetric or asymmetric uncertain parameters with known means and unknown probability distributions, and (ii) bounded uncertain parameters with limited information on their means. With the proposed *a priori* and *a posteriori* bounds, robust counterpart optimization can be applied to models with parameters matching the latter case for the first time. The new *a priori* bounds yield smaller uncertainty sets while guaranteeing the same probability of constraint violation when compared with existing methods. A new *a posteriori* bound for symmetric, unspecified probability distributions can yield lower probabilities of constraint violation for the same robust solution. An *a posteriori* bound given uncertain parameters with limited information on their means is also provided. These methods can provide greatly improved objective function values, both when used within a traditional robust counterpart framework and when applied to the *a priori*–*a posteriori* iterative algorithm of Li and Floudas (2014).

2. Background

The scope of this work includes uncertain parameters which participate linearly or are coefficients of variables participating linearly in a constraint or objective function. Given continuous variables x and integer variables y, consider constraint i with nonlinear function $f_i(x, y)$:

$$f_i(x, y) + \sum_k a_{ik} x_k + \sum_\ell b_{i\ell} y_{i\ell} + \sum_m p_{im} \le 0. \tag{1}$$

Relevant to this work is the case where the exact value of some or all of the parameters a_{ik}, $b_{i\ell}$, and p_{im} are unknown. Without loss of generality, constraint (1) can be reformulated as:

$$\begin{aligned} f_i(x, y) + t_i &\le 0 \\ -t_i + \sum_k a_{ik} x_k + \sum_\ell b_{i\ell} y_{i\ell} + \sum_m p_{im} &\le 0. \end{aligned} \tag{2}$$

The same type of reformulation can be applied to an objective function under uncertainty. Thus, we will assume that all uncertain constraints only involve linearly participating variables which are continuous or integer. The typical presentation of robust optimization goes further and assumes that the entire model is an LP or MILP. We present the background section under this assumption so as to maintain consistency. The study of uncertainty in parameters that participate nonlinearly in the objective function or constraints is beyond the scope of this paper and will be the subject of future work.

The general form of a LP or MILP under uncertainty is as follows:

$$\begin{aligned} \max_{x,y} \; & \sum_k \tilde{c}_k x_k + \sum_\ell \tilde{d}_\ell y_\ell \\ \text{s.t.} \; & \sum_k \tilde{a}_{ik} x_k + \sum_\ell \tilde{b}_{i\ell} y_\ell \le \tilde{p}_i \quad \forall i \\ & y_\ell \in \{0, 1\} \quad \forall \ell \end{aligned} \tag{3}$$