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An explicit finite difference model for simulating weakly nonlinear and weakly dispersive waves over slowly varying water depth

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ABSTRACT

In this paper, a modified leap-frog finite difference (FD) scheme is developed to solve Non linear Shallow Water Equations (NSWE). By adjusting the FD mesh system and modifying the leap-frog algorithm, numerical dispersion is manipulated to mimic physical frequency dispersion for water wave propagation. The resulting numerical scheme is suitable for weakly nonlinear and weakly dispersive waves propagating over a slowly varying water depth. Numerical studies demonstrate that the results of the new numerical scheme agree well with those obtained by directly solving Boussinesq-type models for both long distance propagation, shoaling and re-fraction over a slowly varying bathymetry. Most importantly, the new algorithm is much more computationally efficient than existing Boussinesq-type models, making it an excellent alternative tool for simulating tsunami waves when the frequency dispersion needs to be considered.

1. Introduction

In the past several decades, several numerical models have been developed for calculating transoceanic tsunami propagation. Most of these numerical models are based on the Shallow Water Equations (SWE). Because it is essential to quickly produce numerical results for tsunami early warning system, SWE-based models adopt explicit finite difference schemes (e.g., Liu et al., 1995; Titov et al., 2001). These models have been implemented in tsunami early warning systems and have been successfully applied to several recent tsunami events.

For many tsunami events, the width of a ruptured area (fault plane) is much larger than water depth (e.g., 200 km vs. 3 km in the 2004 Sumatra earthquake (Wang and Liu, 2006)) and the resulting wavelength of the leading tsunami wave is also much larger than the water depth in the deep-ocean. Therefore, the frequency dispersion is indeed negligible during the tsunami propagation phase. However, historically there are tsunami events in which the width of fault plane is relatively small (e.g., 18.3 km in Algerian earthquake in 2003 (Wang and Liu, 2005) and the frequency dispersion plays a role in the determination of the leading wave height. It is also well known that the effects of frequency dispersion are accumulative and become increasingly important as tsunamis travel a long distance (Mei, 1989; Madsen et al., 2008). This becomes an important issue when tsunami's global impacts are assessed (Titov et al., 2005b). Finally, in the state of arts approach for establishing a tsunami early warning system (e.g., Wei et al., 2003; Titov et al., 2005a), the anticipated fault plane is first divided up into many small "fault plane elements". Numerical results for tsunami propagation in the deep-ocean basin corresponding to a prescribed seafloor displacement form on each fault plane element are pre-calculated and stored for tsunami forecasting purpose (Titov et al., 2005a). We note that even if the width of the entire anticipated fault plane is very large in comparison with the water depth, the width of the "fault plane element" might not be so if the spatial inhomogeneity is anticipated. Therefore, the effects of frequency dispersion need to be considered in the pre-calculation process.

On the other hand, tsunami propagation models based on Boussinesq-type (BT) equations are capable of considering frequency dispersive effects from shallow to intermediate water (Nwogu, 1993; Wei and Kirby, 1995; Kirby et al., 1998; Lynett et al., 2002; Lynett and Liu, 2004a,b; Hsiao et al., 2005; Lynett, 2006, 2007; Grilli et al., 2007). However, because of the appearance of higher order terms associated with the frequency dispersion, the algorithms for BT models call for finer spatial and temporal resolution and higher-order numerical algorithms, so that numerical dispersion and truncation errors will not affect the accuracy of numerical solutions. The required finer grids as well as high-order numerical algorithm make BT models much more computationally expensive, not to mention that implicit schemes are usually adopted to solve BT equations for stability reason. Extremely high CPU cost rules out the application of BT equation models for large-scale tsunami simulations.

Imamura et al. (1988) (hereafter IM88) presented a FD model for the simulation of transoceanic tsunamis, which solves the Linear

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Shallow Water Equations (LSWE) using the explicit leap-frog scheme. The frequency dispersion terms neglected in the LSWE are taken into account by utilizing the numerical dispersion inherent in the leap-frog FD scheme. This is done by choosing grid size and time step according to a specified criterion. However, the frequency dispersion effects oblique to the principle axes of the computational domain were not properly represented in the original algorithm and the method was limited to constant water depth. Cho (1995) (hereafter CH95) improved upon IM88's numerical algorithm so that frequency dispersion effects are correctly included in all directions of tsunami propagation. Consequently, the numerical algorithm actually produce numerical results satisfying the traditional Boussinesq equations in a constant water depth.

When the frequency dispersion is important in simulating tsunami propagation over a varying water depth, the frequency dispersion effects need to be carefully considered at every grid point in the entire computational domain. Thus, following the framework of IM88 and CH95, the grid size needs to be locally adjusted according to the time step size and the local water depth, which makes the implementation of these algorithms somewhat difficult. Yoon (2002) developed a new FD scheme that satisfies the local frequency dispersion requirement for a varying water depth while a uniform grid system is still employed. In Yoon's method a hidden moving grid system determined locally from the condition suggested by IM88 is introduced. The physical variables associated with the hidden moving grid system are obtained by interpolating the variables assigned on the actual uniform grid points. Yoon (2002) demonstrated that this scheme provides significant improvements on the frequency dispersion effects compared to those of IM88 and CH95. Chen et al. (2000) also presented a numerical scheme to adjust the numerical dispersion terms and achieved improvements similar to that of Yoon (2002). However the range of adjustment is very limited.

More recently, Yoon et al. (2007) developed another scheme in which the linearized BT equations is resolved with an explicit FD method. The resulting numerical dispersion is again used to improve the physical frequency dispersion. We note that in Yoon et al. (2007) the BT equations were combined into a wave equation in terms of the free surface displacement. A finite element (FE) version of this model is also available in Yoon et al. (2008). Since their models are developed based on linearized Boussinesq equations and the variation of water depth is also taken into account, they show good performance for dispersive waves over variable water depth and the computational efficiency is very high. However, their models are only applicable for linear waves. As tsunamis shoal onto a continental shelf, nonlinearity gradually plays a significant role in the transformation. Linear model is no longer useful. Moreover, since only free surface elevations are solved from the models by Yoon et al. (2007, 2008) and the velocity field must be solved separately, the estimation of the computational efficiency is not necessarily conservative.

In this paper, a modified explicit leap-frog finite difference method is proposed to solve the Nonlinear Shallow Water Equations (NSWE) over a slowly varying water depth. In the new algorithm, with the idea of hidden adjusted grid system proposed by Yoon (2002), the leading-order terms of the numerical truncation errors in solving the NSWE are manipulated to recover the physical frequency dispersion neglected in the NSWE. As a result, the resultant modified shallow water equations together the leading-order truncation error terms due to numerical discretization recovers the classical form of Boussinesq equations. Numerical studies show that the proposed algorithm is capable of adequately simulating evolutions of weakly nonlinear and weakly dispersive waves over a constant or a slowly varying water depth. Most importantly, the proposed model still adopts explicit FD schemes and has a much higher computational efficiency than existing BT models.

2. Governing equations

By introducing three typical length scales: wave amplitude a_0, wavelength l_0 and water depth h_0, the following dimensionless variables can be defined: $\eta = \eta'/a_0$, $(x,y) = (x',y')/l_0$, $H = H'/h_0$, $P = P'/\left(a_0\sqrt{gh_0}\right)$, $Q = Q'/\left(a_0\sqrt{gh_0}\right)$, and $t = t'\sqrt{gh_0}/l_0$, in which t' denotes dimensional time; P' and Q' are the depth-averaged dimensional volume fluxes in the x' and y' directions, respectively; $P = Hu$ and $Q = Hv$ with $H = \epsilon\eta + h$ being the total dimensionless water depth, η the dimensionless free surface elevation and h the dimensionless still water depth. Then, the traditional forms of depth-integrated Boussinesq equations over a varying water depth can be written in the following dimensionless form (CH95)

$$\frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0, \tag{1}$$

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \epsilon\left[\frac{\partial}{\partial x}\left(\frac{P^2}{H}\right) + \frac{\partial}{\partial y}\left(\frac{PQ}{H}\right)\right] + H\frac{\partial \eta}{\partial x}\\ &= \mu^2\left\{\frac{h^2}{2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial t}\right)\right] - \frac{h^3}{6}\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x \partial t}\left(\frac{P}{h}\right) + \frac{\partial^2}{\partial y \partial t}\left(\frac{Q}{h}\right)\right]\right\}\\ &\quad + O\left(\mu^4, \epsilon\mu^2\right),\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial Q}{\partial t} &+ \epsilon\left[\frac{\partial}{\partial x}\left(\frac{PQ}{H}\right) + \frac{\partial}{\partial y}\left(\frac{Q^2}{H}\right)\right] + H\frac{\partial \eta}{\partial y}\\ &= \mu^2\left\{\frac{h^2}{2}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial t}\right)\right] - \frac{h^3}{6}\frac{\partial}{\partial y}\left[\frac{\partial^2}{\partial x \partial t}\left(\frac{P}{h}\right) + \frac{\partial^2}{\partial y \partial t}\left(\frac{Q}{h}\right)\right]\right\}\\ &\quad + O\left(\mu^4, \epsilon\mu^2\right).\end{aligned} \tag{3}$$

Here, two non-dimensional parameters, ϵ and μ, are introduced and defined as

$$\epsilon = \frac{a_0}{h_0}, \quad \mu = \frac{h_0}{l_0}, \tag{4}$$

which measure the nonlinearity and frequency dispersion, respectively. For the non-dispersive wave system ($\mu = 0$) the phase speed is independent of wave number (or wave frequency). In a constant depth, the wave form of a non-dispersive plane wave remains the same, provided that the nonlinearity is very small. However, if the nonlinearity becomes significant, the phase speed increases as the free surface elevation increases, causing the steepening of the wave front. In contrast, when the frequency dispersion effects are significant, each wave component (with different wave frequency) propagates with different phase speed, resulting in the degeneration (spreading or dispersion) of wave form. In the traditional Boussinesq equations, $\epsilon = O(\mu^2)$ is assumed.

Substituting Eq. (1) into the terms, associated with μ^2, on the right-hand side of Eqs. (2) and (3) so as to eliminate P and Q from the terms, we obtain the following new forms of momentum equations:

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \epsilon\left[\frac{\partial}{\partial x}\left(\frac{P^2}{H}\right) + \frac{\partial}{\partial y}\left(\frac{PQ}{H}\right)\right] + H\frac{\partial \eta}{\partial x}\\ &= -\mu^2\frac{h^3}{3}\left[\frac{\partial^3 \eta}{\partial x^3} + \frac{\partial^3 \eta}{\partial x \partial y^2}\right]\\ &\quad -\mu^2\frac{h^2}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial \eta}{\partial y}\right) + \frac{\partial h}{\partial x}\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right)\right] + O\left(\mu^4, \epsilon\mu^2\right),\end{aligned} \tag{5}$$