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Note

Multi-objective optimization with convex quadratic cost functions: A multi-parametric programming approach

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ABSTRACT

In this note we present an approximate algorithm for the explicit calculation of the Pareto front for multi-objective optimization problems featuring convex quadratic cost functions and linear constraints based on multi-parametric programming and employing a set of suitable overestimators with tunable suboptimality. A numerical example as well as a small computational study highlight the features of the novel algorithm.

1. Introduction

We consider the following multi-objective optimization (MOO) problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \{f_1(x), f_2(x), \ldots, f_N(x)\} \\ \text{subject to} \quad & Ax \le b, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and the objective functions are of the form

$$f_i(x) = x^T Q_i x + c_i^T x + d_i, \quad \forall i = 1, \ldots, N, \tag{2}$$

where $Q_i \in \mathbb{R}^{n\times n} \succ 0$, $c_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$, $\forall i = 1, \ldots, N$. Problems of type (1) arise in many different applications, such as engineering, economics and biological systems (Miettinen, 1998; Copado-Méndez et al., 2014; Mian et al., 2015). A solution x^* of problem (1) is thereby called optimal if it is a Pareto point.

Definition 1 *(Pareto point and Pareto front).* A point x^* is called a Pareto point if there does not exist a point $\hat{x}$ such that there exists $f_i(\hat{x}) < f_i(x^*)$ and $f_j(\hat{x}) \le f_j(x^*)$, $j \ne i$. The set of all Pareto points is called the Pareto front $\mathcal{P}$.

One of the most well known strategies to obtain a Pareto point is the ϵ-constraint method,[1] i.e.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_1(x) \\ \text{subject to} \quad & f_j(x) \le \epsilon_j, \quad \forall j = 2, \ldots, N \\ & Ax \le b \\ & x \in \mathbb{R}^n, \end{aligned} \tag{3}$$

where the parameter ϵ_j denotes an upper bound on the function $f_j(x)$. Another important strategy is the linear scalarization method,[2] i.e.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{i=1}^{N} w_i f_i(x) \\ \text{subject to} \quad & Ax \le b, \\ & w_i \ge 0, \forall i = 1, \ldots, N, \\ & \sum_{i=1}^{k} w_i = 1, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{4}$$

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[1] Note that the choice of $f_1(x)$ as the objective function is arbitrary (see Miettinen, 1998).

[2] This method is sometimes also referred to as weighting method (Miettinen, 1998).

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However in both strategies the solution of the MOO problem depends on the values of certain parameters, namely ϵ_j and w_i. Hence, while many researchers consider the iterative solution of the resulting optimization problems for different parameter values (Capitanescu et al., 2015; Antipova et al., 2015), some attention has been given to the explicit calculation of the entire Pareto front *via* parametric programming[3], which solves optimization problems as a function and for a range of certain parameters. In (Gass and Saaty, 1955; Yuf and Zeleny, 1976) the authors consider the case of linear cost functions, and in (Papalexandri and Dimkou, 1998) the case of a mixed-integer nonlinear MOO was considered. The case of quadratic cost functions was treated in (Goh and Yang, 1996; Ghaffari-Hadigheh et al., 2010), although either only conceptually or for the case where the quadratic part remains constant. Thus in this note, we propose an algorithm for the approximate explicit solution of MOO problems with general convex quadratic cost functions and linear constraints by multi-parametric programming.

2. Multi-objective optimization via multi-parametric programming

By inspection, it is clear that problem (1) results in a quadratically constrained quadratic programming (QCQP) problem, whose explicit solution would require the solution of a multi-parametric QCQP (mp-QCQP) for which no efficient solution approach exists.[4] In this section we present an algorithm to approximate the original mp-QCQP using a multi-parametric quadratic programming (mp-QP) with a suitable set of affine overestimators, which can be readily solved with existing solvers.

2.1. Reformulation of mp-QCQP

In order to convert the mp-QCQP problem (3) into a mp-QP, given a convex quadratic function $f(x)$, a suboptimality gap ϵ and a domain $\mathcal{X} = \{x \in \mathbb{R}^n | Ax \le b\}$, the aim is to find a suitable convex piecewise affine overestimator $\mathcal{F}(x) = \max\limits_{1 \le k \le M} \{\bar{f}_1(x), \bar{f}_2(x), \ldots, \bar{f}_M(x)\}$, such that

$$0 \le \mathcal{F}(x) - f(x) \le \epsilon \tag{5}$$

where

$$\bar{f}_k(x) = a_k^T x + b_k, \quad \forall k = 1, \ldots, M. \tag{6}$$

Remark 1. It is well known that $\mathcal{F}(x)$ can be described via a set of linear inequalities (Boyd and Vandenberghe, 2004).

First, we consider a point x_R and define

$$f(x) \ge f(x_R) + \nabla f(x_R)(x - x_R) \tag{7}$$

based on the first-order Taylor expansion. Given a suboptimality gap ϵ, it is obvious that the neighbourhood around x_R for which Eq. (7) is a sufficient approximation is given by

$$f(x) - f(x_R) - \nabla f(x_R)(x - x_R) \le \epsilon. \tag{8}$$

Substitution of $f(x) = x^T Qx + c^T x + d$ and $\nabla f(x) = 2Qx + c$ then yields

$$x^T Qx - 2x_R^T Qx + x_R^T Qx_R \le \epsilon. \tag{9}$$

[3] In the following, this is referred to as the explicit solution of a MOO problem.

[4] An exact algorithm has been derived for the single parameter case in (Ritter, 1966). Classically, the explicit solution of the Pareto front is approximated by solving a set of global optimization problems (Marler and Arora, 2004).

Thus, in order to ensure that Eq. (9) holds over the entire domain $\mathcal{X}$, it is necessary to find a set of points x_R^i, $1, \ldots, M$ such that

$$\begin{aligned} \epsilon \ge \eta = \quad & \max_x \quad \min_i \quad x^T Qx - 2x_R^{i,T} Qx + x_R^{i,T} Qx_R^i \\ & \text{s.t.} \qquad\qquad x \in \mathcal{X}. \end{aligned} \tag{10}$$

Note that problem (10) can be reformulated into a classical min–max problem via

$$\max_x . \ \min_i . \ F_i(x) = -\min_x . \ \max_i .(-F_i(x)), \tag{11}$$

for which commercial solvers are readily available (e.g. in the MATLAB® Optimization Toolbox). Thus, we obtain

$$\mathcal{F}(x) = \max_{1 \le k \le M} \{\bar{f}_1(x), \bar{f}_2(x), \ldots, \bar{f}_M(x)\} \tag{12}$$

with

$$\bar{f}_i(x) = f(x_R^i) + \nabla f(x_R^i)(x - x_R^i) + \epsilon, \quad \forall i = 1, \ldots, M. \tag{13}$$

Remark 2. As problem (10) is non-convex, the convexity assumption for the objective functions in Eq. (2) is not necessary for the application of the general strategy outlined in this paper. As however Eq. (7) only holds for convex $f(x)$, it is necessary to choose a set of affine overestimators which do not require a convex objective function such as the McCormick relaxations (McCormick, 1976).

The algorithm on how to calculate $\mathcal{F}(x)$ is presented in Algorithm 1.

Algorithm 1. Piecewise-affine approximation on $f(x)$

```
Require: f(x), 𝒳, ϵ
Output: 𝓕(x)
1:   Set η ← ∞, x_R^1 = argmin {problem (14)}, M = 1
2:   while η > ϵ do
3:     Solve problem (10)
4:     if η > ϵ then
5:       Set M = M + 1, x_R^M = argmin{problem (10)}
6:     end if
7:   end while
8:   Set 𝓕(x) = {f(x_R^i) + ∇f(x_R^i)(x − x_R^i),   ∀i = 1, ..., M}
```

2.2. Remarks

Initial point x_R^1: The initial point of the algorithm is the Chebyshev center of the polytope $\mathcal{X}$, which is calculated via

$$\begin{aligned} \underset{x,t}{\text{minimize}} \quad & -t \\ \text{subjectto} \quad & Ax \le (b - \|A\|_2 t) \\ & x \in \mathcal{X} \subset \mathbb{R}^n, \quad t \in \mathbb{R}, \end{aligned} \tag{14}$$

where $\|\cdot\|_2$ denotes the 2-norm.

Choice of ϵ: Obviously, the complexity of the approximation as well as the quality of the solution of the mp-QP problem depend on the choice of ϵ. As ϵ denotes the absolute suboptimality, it cannot be fixed without considering the objective function $f(x)$. The reason relative suboptimality is not used as a measure for the quality of the approximation is that it favours very tight approximation around the origin while looser approximations further off. In order to avoid this distortion, within this paper we define the following relation:

$$\epsilon = \epsilon^* \max_x |f(x)|, \tag{15}$$