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Robust optimization under correlated uncertainty: Formulations and computational study

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ABSTRACT

The uncertainty set-induced robust optimization framework has received considerable attention in the past decades. It has been extensively studied in literature and applied to address various decision-making problems. However, existing robust optimization methods generally assume that the uncertain parameters are independent. As a result, the traditional robust optimization methods may lead to a conservative solution in practice when correlations between uncertain parameters exist. In this work, we present novel results on robust optimization under correlated uncertainties that appear in a single constraint. Robust counterpart optimization formulations are derived based on various types of uncertain sets. Numerical and application examples are studied to compare the performance of robust optimization by incorporating various levels of correlation. The results demonstrate that incorporating more accurate correlation into the robust optimization formulation can lead to less conservative robust solution.

1. Introduction

In deterministic optimization problems, the model parameters are assumed to be known with certain values. However, in real applications many realistic parameters are subject to uncertainty. Under such situations, assuming deterministic values for uncertain parameters will lead to infeasible or suboptimal decisions for practical implementation. Thus, incorporation of parameter uncertainty is necessary in many practical optimization problems. Various approaches have been proposed in the past to address uncertainty in optimization problems, such as robust optimization, stochastic programming with recourse, and chance constraints. Among the various methods, robust optimization addresses the parameter uncertainty based on an uncertainty set, which covers part or the whole region of the uncertainty space. The target of robust optimization is to select the best solution that remains feasible for any realizations of the uncertain parameters in the uncertainty set. Compared to other methods for addressing uncertainty in optimization problems, one of the significant advantages of robust optimization is the computational tractability. The robust counterpart generally does not increase much in model size compared to the deterministic model, and the convexity of the constraint can also be preserved.

The uncertainty set-induced robust optimization framework has been investigated by many studies in past decades and it has been applied to various decision-making problems. One of the earliest work by Soyster (1973) studying robust optimization considered simple perturbations in the data and reformulated the original linear programming problem so that the solution would be feasible under all possible perturbations. However, the approach is very conservative since it ensures feasibility for all potential realizations of the uncertainty. Robust optimization received more attention since the 1990s. El-Ghaoui and Lebret (1997) studied least-squares problems with uncertainty. El-Ghaoui et al. (1998) investigated uncertain semidefinite problems with robust optimization framework. Ben-Tal and Nemirovski (1998, 1999) pointed out that robust formulation becomes a conic quadratic problem for a linear constraint with ellipsoidal uncertainty set. A number of valuable formulations and applications in linear programming and general convex programming have been proposed by Ben-Tal and Nemirovski (2000, 2001). Ben-Tal et al. (2004) proposed an approach for linear programming problems where some of the decision variables must be determined before the realization of uncertain data, while the other decision variables can be set after realization. Bertsimas and Sim (2004) derived a robust formulation for uncertain linear programming problems using budget parameter, which can control the degree of conservatism of the solution. Bertsimas et al. (2004) studied the robust counterpart of linear programming problems based on the uncertainty set defined by a general norm. By generalizing the symmetric uncertainty sets,

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Chen et al. (2007) investigated asymmetrical set induced robust optimization.

Besides the above contributions made by the operations research community, robust optimization also received studies by the process systems engineering researchers. Li and Ierapetritou (2008) studied the application of various robust optimization formulations to process scheduling problem under uncertainty. Lin et al. (2004) and Janak et al. (2007) studied robust optimization for mixed integer linear optimization problems with uncertainty under both bounded and several known probability distributions. Verderame and Floudas (2009) applied the robust optimization framework to operational planning problems. Li et al. (2011) presented a systematic study on the set-induced robust counterpart optimization for both linear optimization and mixed integer linear optimization problems. Robust counterpart formulations were derived based on different types of uncertainty set. While those work focused on robust optimization formulations for an individual constraint, Yuan et al. (2015) proposed robust optimization formulations that can be used to approximate joint chance constraints in a recent work.

The issue of robust solution quality also received attention in several recent works. To improve the solution quality of a robust optimization problem, the main issue is to find an appropriate uncertainty set size. To be more specific, a small set size is preferred, while the solution reliability is met, because it leads to less conservative solution. The traditional way of determining the set size is based on the a priori probability bound, which is a function of the set size. Li et al. (2012) proposed various a priori and a posteriori probability bounds. Based on that, Li and Floudas (2014) proposed an iterative method to improve the robust solution quality by iteratively adjusting the set size until the probability of constraint satisfaction reaches the desired level. In another work, Li and Li (2015) proposed a method to identify the smallest set size with the least conservative solution through an optimal set size identification algorithm.

Although considerable progress has been made in the area of robust optimization, in the existing methods, independence is generally assumed among uncertainties in the parameters. However, in practice, correlations may arise in the uncertainties. For instance, the price and demand of crude oil are correlated, which may affect the refinery planning decision making. As shown by the computational studies of this work, the traditional robust optimization methods that ignore the correlation may lead to a conservative solution. Hence, it is of great importance to consider the correlation among uncertainties in robust optimization.

In this work, novel results are presented for robust optimization under correlated uncertainties. First, the robust optimization framework is proposed for correlated uncertainty within a single constraint. Based on five different types of uncertainty set, the corresponding robust counterpart optimization formulations are derived. Specifically, for unbounded and correlated uncertainties, box, polyhedral, and ellipsoidal type of uncertainty sets are selected to derive the set-induced robust optimization formulations, and for bounded correlated uncertainties, "interval + polyhedral" and "interval + ellipsoidal" types of uncertainty set are applied to derive the set-induced robust optimization formulations. Finally, numerical and application examples are studied to investigate the proposed robust optimization framework for correlated uncertainty within a constraint. Different levels of the correlations are considered to demonstrate the necessity of incorporating uncertainty correlations into the traditional robust optimization approach.

The rest of this paper is organized as follows: In Section 2, the robust counterpart formulations for uncertainty within a single constraint are derived based on five different types of uncertainty set. In Section 3, numerical examples are studied to demonstrate the effectiveness of the proposed robust counterpart formulations, and to investigate the effect of correlation modeling in robust optimization. The proposed method is further applied to a production planning example in Section 4. Finally, the paper is concluded in Section 5.

2. Robust counterpart optimization formulations

Consider the following linear optimization problem with uncertain constraint coefficients:

$$\begin{aligned} \max_{x \in X} \quad & cx \\ s.t. \quad & \tilde{a}^T x \le b \end{aligned} \tag{1}$$

where $x \in R^n$ represents the decision variables, $\tilde{a}$ is an $n \times 1$ vector with entries $\tilde{a}_j, j = 1, \ldots, n$, i.e., $\tilde{a} = [\tilde{a}_1, \ldots, \tilde{a}_n]^T$, and n is the number of decision variables. Without loss of generality, the uncertainties in the coefficients can be modeled as:

$$\tilde{a}_j = a_j + u_j,\ j = 1, \ldots, n \tag{2}$$

where a_j is the nominal (most expected) value of $\tilde{a}_j$, and u_j is the random part following a distribution with zero mean. Separating the deterministic part and the uncertain part, the constraint becomes:

$$a^T x + u^T x \le b \tag{3}$$

where $u = [u_1, \ldots, u_n]^T$. To ensure the constraint satisfaction under the worst-case scenario of an uncertainty set U, the constraint with uncertainty is rewritten as:

$$a^T x + \max_{u \in U} u^T x \le b \tag{4}$$

The above constraint (4) is the robust counterpart of the uncertain constraint (3). Its explicit expression depends on the uncertainty set U. Robust counterpart optimization problem of (1) can be obtained by replacing the uncertain constraint with its robust counterpart. In this work, five types of uncertainty set including box, ellipsoid, polyhedral, interval + ellipsoidal, and interval + polyhedral. The design of uncertainty set is related to the distribution of the uncertainty. If the uncertainty is subject to unbounded distribution, then the box, ellipsoidal, and polyhedral type of uncertainty set is appropriate to be used in the robust optimization framework, where the size of the uncertainty set is not restricted. Instead, if the uncertainty is subject to bounded distribution, then the bounds information can be involved in the uncertainty set as intervals to avoid an unnecessarily large uncertainty set (which will lead to more conservative solution). Hence, the interval + ellipsoidal uncertainty set and the interval + polyhedral uncertainty set are appropriate for bounded uncertainty distribution. In the following subsections, explicit robust counterpart constraint formulations of (4) will be investigated based on different uncertainty sets.

2.1. *Robust formulations for bounded uncertainty distribution*

Property 1 (Box uncertainty set). The robust counterpart optimization formulation for constraint (4) under the box-type uncertainty set $U_{box} = \left\{ u \,\middle|\, \left\| Mu \right\|_\infty \le \Delta \right\}$ is:

$$\begin{cases} a^T x + \Delta \sum\limits_{k=1}^{n} y_k \le b \\ -y_k \le \sum\limits_{j=1}^{n} m_{kj} x_j \le y_k, \quad \forall k = 1, \ldots, n \end{cases} \tag{5}$$