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Lattice Boltzmann method for Casimir invariant of two-dimensional turbulence*

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Abstract: The Casimir invariants of the 2-D turbulence are investigated by the lattice Boltzmann method. A coarse-graining approach is used, that allows to resolve the flux of the Casimir invariant in scale and in space. It is found that the flux of the enstrophy cascades to small scales and the direction cascade of the energy flux is upscaled. Moveover, the probability distribution function (PDF) of the enstrophy flux gives a clear evidence that the enstrophy cascades to smaller scales. Finally, the behavior of the cascade of the high-order Casimir invariants Z^n is discussed. The flux of the fourth-order Casimir invariant Z^4 cascades to small scales. The flux of Z^n has a logarithmic relationship with the scale, that is, $\Pi_l^{Z^n} \sim l^{\zeta_n} (n = 2,4,6)$.

Key words: 2-D turbulence, Casimir invariants, lattice Boltzmann method

Introduction

It is commonly believed that the simultaneous conservation of the energy and the enstrophy by the advection term of the forced 2-D Navier-Stokes equations gives rise to a dual turbulence cascade when the Reynolds number tends to infinity[1-3]. Under statistically stationary conditions, when the turbulent flow is sustained by an external forcing acting in a typical force scale l_f, a double cascade develops. According to the Kraichnan theory, at a large scale, i.e., when the wave numbers $k \ll k_f \sim l_f^{-1}$, the energy spectrum assumes the form $E(k) \approx \varepsilon^{2/3} k^{-5/3}$ while in small scales, $k \gg k_f$, the prediction is $E(k) \approx \eta^{2/3} k^{-3}$, with a possible logarithmic correction[1]. Here $\eta = k^2 \varepsilon$. ε and η are, respectively, the energy and the enstrophy injection rates.

In addition to conserving the energy and the enstrophy, the nonlinear terms of the 2-D incompressible Navier-Stokes equation are well known to conserve the global integral of any continuously differentiable function of the scalar vorticity field, which are known as the Casimir invariants. A fundamental question is whether these Casimir invariants also play an underlying role in the turbulence cascade, in addition to the rugged quadratic invariants (the enstrophy). Whether they cascade to large or small scales is an open question. Polyakov' minimal conformal field theory model suggests that the higher-order Casimir invariants cascade to large scales[4], while Eyink[5] predicted that they might instead cascade to small scales. Bowman[6] pointed out that the fourth power of the vorticity cascades to small scales by using the well-resolved implicitly dealiased pseudospectral simulations. Meanwhile, this study raises the question of whether the Kraichnan theory of the unbounded 2-D turbulence, based solely on the uniform flux of the energy in large scales and that of the enstrophy in small scales, needs to be re-examined to account for a direct cascade of the Casimir invariants to smaller scales.

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A better understanding of the physical mechanism on the basis of the cascades can be obtained by looking at the distribution of the fluxes of the Casimir invariant in scales. Here the key analysis method we use is a "coarse-graining" or "filtering" approach for analyzing the scale interactions in complex flows. Eyink[7] developed the formalism mathematically to analyze the fundamental physics of the scale coupling in turbulence, which was laterly applied to numerical and experimental studies of flows of 2-D turbulence[8-12] . For any field $a(x)$, a "coarse-graining" or "filtering" field, which contains modes at a length-scale $> l$, is defined as

$$a_l(x) = \int d^3 r G_l(r) a_l(l+x),$$

where $G_l(r)$ is a normalized convolution kernel. It is well known that the lattice Boltzmann method (LBM) is valid in the investigations of 2-D turbulence[3,13,14]. In this paper, this "filtering" approach is used to investigate the flux of the Casimir invariant in the frame of the LBM.

1. Preliminaries

1.1 *The flux of Casimir invariants*

The balance equations governing the local conservation of the vorticity invariants are expressed in space and in scale. Due to the viscous effect, the high order Casimir invariants are generally not in conservation. However, it is verified that the viscosity has no influence on the definition of the flux of high order Casimir invariants. To introduce the concepts in the simple context, we discuss first the free evolution, i.e., the equations without any external forcing. Thus, our starting point is the 2-D Euler equations in the "vorticity formulation".

$$\partial_t \omega(r,t) + \nabla \cdot [u(r,t)\omega(r,t)] = 0 \tag{1}$$

That is, we consider the large-scale vorticity defined by convolution $\bar{\omega}_l = G_l \omega$ and the large-scale velocity defined by $\bar{u}_l = G_l u$, where G_l is taken to be the Gaussian filter. If the filter is convoluted with the equation of motion, Eq.(1), an equation for the large-scale vorticity field is obtained

$$\partial_t \bar{\omega}_l(r,t) + \nabla \cdot [\bar{u}_l(r,t)\bar{\omega}_l(r,t) + \sigma_l(r,t)] = 0 \tag{2}$$

where σ_l is the space transport of the vorticity due to the eliminated small-scale turbulence. From Eq.(2), a balance equation is derived for the local density $h_l(r,t) = 0.5\bar{\omega}_l^2(r,t)$,

$$\partial_t h_l(r,t) + \nabla \cdot K_l(r,t) = -\Pi_l^Z(r,t) \tag{3}$$

where $K_l(r,t)$ represents the space transport of the large-scale enstrophy,

$$K_l(r,t) = h_l(r,t)\bar{u}_l(r,t) + \bar{\omega}(r,t)\sigma_l(r,t) \tag{4}$$

and $\Pi_l^Z(r,t)$ is the enstrophy flux out of large scale modes into small-scale modes,

$$\Pi_l^Z(r,t) = -\nabla\bar{\omega}(r,t)\sigma_l(r,t) \tag{5}$$

In Eq.(3), we see that in order for $\Pi_l^Z(r,t) > 0$ to have a net positive value, the turbulence vorticity transport $\sigma_l(r,t)$ should tend to be antiparallel to the large-scale vorticity gradient $\nabla\bar{\omega}(r,t)$. The required statistical anticorrelation between $\sigma_l(r,t)$ and $\nabla\bar{\omega}(r,t)$ is an alignment property characteristic of the enstrophy cascade. It is analogous to the much-studied alignment of the stress tensor τ_l due to small scales and the large-scale strain $\bar{S}_l$, which underlies the energy cascade to small scales in 3-D.

An identical analysis can be made of the balance for the local densities $h_l^n(r,t) = 0.5\bar{\omega}_l^n(r,t)$ of the contribution to the Casimir invariants Z^n in the large-scale modes $(Z^n = (1/2)\langle \omega^n \rangle)$ [7]. By a similar calculation as before, it follows that

$$\partial_t h_l^n(r,t) + \nabla \cdot K_l^n(r,t) = -\Pi_l^{Z^n}(r,t) \tag{6}$$

where $K_l^n(r,t)$ represents the space transport of the large-scale Z^n,

$$K_l^n(r,t) = h_l^n(r,t)\bar{u}_l(r,t) + h_l^{n-1}(r,t)\sigma_l(r,t) \tag{7}$$

and $\Pi_l^{Z^n}(r,t)$ is the flux of Z^n out of large scale modes into small-scale modes,

$$\Pi_l^{Z^n}(r,t) = h_l^n(r,t)\,\Pi_l^Z(r,t) \tag{8}$$

It is of some interest that it is simply proportional to the enstrophy flux itself, when $n > 2$.

1.2 *lattice Boltzmann method (LBM)*

The Navier-Stokes equation for the fluid flows can be simulated by the LBM in a simple and efficient way[13,15-19]. The LBM has its roots in the kinetic theory, and the general idea behind this scheme is to compute a probability distribution function $f_i(r,t)$, where