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Optimal robust optimization approximation for chance constrained optimization problem

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ABSTRACT

Chance constraints are useful for modeling solution reliability in optimization under uncertainty. In general, solving chance constrained optimization problems is challenging and the existing methods for solving a chance constrained optimization problem largely rely on solving an approximation problem. Among the various approximation methods, robust optimization can provide safe and tractable analytical approximation. In this paper, we address the question of what is the optimal (least conservative) robust optimization approximation for the chance constrained optimization problems. A novel algorithm is proposed to find the smallest possible uncertainty set size that leads to the optimal robust optimization approximation. The proposed method first identifies the maximum set size that leads to feasible robust optimization problems and then identifies the best set size that leads to the desired probability of constraint satisfaction. Effectiveness of the proposed algorithm is demonstrated through a portfolio optimization problem, a production planning and a process scheduling problem.

1. Introduction

Chance constraint (also called probabilistic constraint) is an important tool for modeling reliability on decision making in the presence of uncertainty. A general chance constrained optimization problem takes the following form:

$$\begin{aligned} &\max_{x\in X} f(x) \\ &\text{s.t.} \quad \Pr\{h(x,\xi)\le 0\}\ge 1-\alpha \end{aligned} \tag{1}$$

where x represents the decision variables, ξ denotes the uncertain parameters, α is a reliability parameter representing the allowed constraint violation level ($0<\alpha<1$). The chance constraint $\Pr\{h(x,\xi)\le 0\}\ge 1-\alpha$ enforces that the constraint $h(x,\xi)\le 0$ is satisfied with probability $1-\alpha$ at least (or violated with probability α at most).

The chance constrained optimization problem was introduced in the work of Charnes et al. (1958) and an extensive review can be found in Prékopa (1995). There are many challenging aspects of solving chance constrained optimization problem. Except for a few specific probability distributions (e.g., normal distribution), it is difficult to formulate an equivalent deterministic constraint for the chance constraint. Furthermore, checking the feasibility of a chance constraint is not easy and the feasible region of chance constrained optimization problem is often nonconvex. To avoid the above difficulties, existing methods for solving a chance constrained optimization problem largely rely on solving an approximation problem. Generally, there are two types of approximation methods used in literature to approximate a chance constraint: sampling based approach and analytical approximation approach.

For the sampling based approach, random samples are drawn from the probability distribution of the uncertain parameters and they are further used to approximate the chance constraint. Scenario approximation and sample average approximation are two different ways of sampling based methods. For scenario approximation, the idea is to generate a set of samples $\xi^1, \xi^2, \ldots, \xi^K$ of the random parameters ξ and approximate the chance constraint with a set of constraints $h(x,\xi^k)\le 0$, $k=1,\ldots,K$. The scenario approximation itself is random and its solution may not satisfy the chance constraint. Research contributions in this direction have been made by Calafiore and Campi (2006), Nemirovski and Shapiro (2006b), Pagnoncelli et al. (2009), etc. For sampling average approximation, it uses an empirical distribution associated with random samples to replace the actual distribution, which is further used to evaluate the chance constraint. This kind of method for chance constrained problems has been investigated by Luedtke and Ahmed (2008), Atlason et al. (2008), Pagnoncelli et al. (2009), etc.

Analytical approximation approach transforms the chance constraint into a deterministic counterpart constraint. Compared to the scenario based approximation, analytical approximation

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provides safe approximation and the size of the model is independent of the required solution reliability. Deterministic approximation of chance constraints can be derived from Chebyshev's inequality (Chebyshev, 1867), Bernstein inequality (Bernstein, 1937), Hoeffding's inequality (Hoeffding, 1963), etc. Nemirovski and Shapiro (2006a) investigated convex approximations of chance constraints. Hong et al. (2011) proposed convex approximations for joint chance constrained programs.

Robust optimization (RO) provides another way for analytically approximating a chance constraint. Robust optimization often requires only a mild assumption on probability distributions, and it provides a tractable approach to obtain a solution that remains feasible in the chance constrained problem. Hence, robust optimization has been widely used to construct a safe approximation of chance constraints. One of the earliest papers on robust counterpart optimization is the work of Soyster (1973). The framework of robust counterpart optimization is also studied by Ben-Tal and Nemirovski (1999, 2000), El Ghaoui and Lebret (1997), El Ghaoui et al. (1998), and Bertsimas and Sim (2004). Ben-Tal and Nemirovski (2000) considered the robust optimization for linear programs under some certain situations, and Lin et al. (2004) and Janak et al. (2007) extended it to mixed integer linear optimization problem. An extensive study is conducted by Li et al. (2011). They studied the robust counterpart optimization techniques for linear optimization and mixed integer linear optimization problems. The *a priori* and *a posteriori* probability bounds on constraint violation/satisfaction were studied by Li et al. (2012), which are further used to improve the solution quality of robust optimization based approximation within an iterative framework (Li and Floudas, 2014).

Robust optimization can provide a safe approximation to the chance constrained problem. However, the quality of the approximation has not received attention in existing literature. A safe approximation can be unnecessarily conservative and lead to a solution that is of poor performance in practice. In this paper, we study the question what is the best safe approximation to an individual chance constraint with robust optimization method. We analyzed the relationship between the size of the uncertainty set and the solution reliability of robust optimization problems. From a motivating example, we discovered the fact the reliability of a robust solution (probability of constraint satisfaction) is not necessarily a monotonic function of the uncertainty set's size, which is contrary to the general intuitive that people tend to have. To address the optimal approximation problem, we proposed a two-step algorithm. The first step is to identify the maximum size that makes the robust optimization problem feasible. The second step is to identify the optimal (smallest) set size that will satisfy the desired solution reliability. We demonstrate the algorithm through a chance constrained portfolio optimization problem, a production planning and a process scheduling problem.

The rest of the paper is organized as follows. In Section 2, the basic robust optimization theory is presented, including the uncertainty set induced robust counterpart optimization constraint and the *a priori* probability upper bound for constraint violation. In Section 3, the robust optimization approximation method for chance constraint optimization problem is introduced and improvement methods are presented based on the *a posteriori* probability bound. In Section 4, a novel method for identifying the optimal set size for robust optimization approximation is proposed. Methods for quantifying the reliability and optimality of a solution to chance constrained optimization problem are introduced in Section 5. The proposed methods are investigated through case studies in Section 6 and the paper is concluded in Section 7.

2. Robust optimization

In this paper, linear constraint under uncertainty is investigated. Consider the following optimization problem with parameter uncertainty:

$$\max_{x\in X} cx \quad \text{s.t.} \quad \sum_j \tilde{a}_j x_j \le b. \tag{2}$$

where the constraint coefficients $\tilde{a}_j$ are subject to uncertainty. Define the uncertainty as $\tilde{a}_j = a_j + \xi_j\hat{a}_j$, $\forall j \in J$, where a_j represent the nominal value of the parameters, $\hat{a}_j$ represent positive constant perturbations, ξ_j represent independent random variables which are subject to uncertainty and J represents the index subset that contains the variables whose coefficients are subject to uncertainty. Constraint in (2) can be rewritten by grouping the deterministic part and the uncertain part as follows:

$$\sum_j a_j x_j + \sum_{j\in J} \xi_j \hat{a}_j x_j \le b. \tag{3}$$

In the set induced robust optimization method, the aim is to find solutions that remain feasible for any ξ in the given uncertainty set U with size Δ so as to immunize against infeasibility. The corresponding robust optimization problem is

$$\max_{x\in X} cx \quad \text{s.t.} \quad \sum_j a_j x_j + \max_{\xi\in U(\Delta)} \left\{ \sum_{j\in J} \xi_j \hat{a}_j x_j \right\} \le b \tag{4}$$

The formulation of the robust counterpart optimization problem is connected with the selection of the uncertainty set U. Based on the work of Li et al. (2011), different robust counterpart optimization formulations can be developed depending on the type of uncertainty set. For example, the box uncertainty set $U_\infty = \{\xi \big| \, |\xi_j| \le \Psi, \ \forall j \in J\}$ induced robust counterpart optimization constraint is given by:

$$\sum_j a_j x_j + \Psi \sum_{j\in J} \hat{a}_j |x_j| \le b \tag{5}$$

And the ellipsoidal uncertainty set $U_2 = \{\xi \big| \sum_{j\in J} \xi_j^2 \le \Omega^2\}$ induced robust counterpart optimization constraint is:

$$\sum_j a_j x_j + \Omega \sqrt{\sum_{j\in J} \hat{a}_j^2 x_j^2} \le b \tag{6}$$

where Ψ and Ω are the size of the box and ellipsoidal uncertainty set, respectively.

For the same type of uncertainty set, as the set size increases, the optimal objective of the robust optimization problem (4) will decrease (for a maximizing objective) because the feasible region of the robust optimization problem (4) becomes smaller. This is shown in the following example.