Computers and Chemical Engineering 70 (2014) 122–132

Contents lists available at ScienceDirect

Computers and Chemical Engineering

journal homepage: www.elsevier.com/locate/compchemeng

Computational complexity and related topics of robustness margin calculation using μ theory: A review of theoretical developments

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ARTICLE INFO

Article history:
Received 15 January 2013
Accepted 20 September 2013
Available online 5 October 2013

Keywords:
Mathematical systems theory
Robust control
Computational complexity

ABSTRACT

This paper provides a comprehensive overview of research related to computational complexity of structured singular value (a.k.a. μ) problems. A survey of computational complexity results in μ problems is followed by a concise introduction to computational complexity theory that is useful to characterize the inherent difficulty of solving an optimization. Results on the study for NP-hardness of ϵ-approximation of μ problems are discussed and conservatism of convex μ upper-bounds including ones obtained from absolute stability theory is studied. NP-hardness of μ computation and conservatism of convex upper-bounds open new research trends. In particular, we give an overview of polynomial-time model reduction methods and probabilistic randomized algorithms that have been active research topics since the mid-1990s.

1. Introduction

The structured singular value μ has been used to study robust stability and performance of dynamic systems with real parametric and dynamic uncertainties. Pioneer works by Doyle (1982) and Safonov (1982) led persistent research interest in computations of robust stability margin and achievable robust performance indices. In robust stability and performance analysis based on structured singular value theory, all possible variations in the plant behavior are considered within a specific structure and the deterministic worst-case scenario determines the values of μ and robustness margin[1] that is the inverse of μ. In this framework for computing the robustness margin of multivariable systems, the value of μ quantifies the bounds of uncertainties that are simultaneously expanded until the system first becomes unstable for some uncertainty from the set.

Historically, Horowitz (1963) proposes a graphical approach that maps the bounded parameteric uncertainty domain into Nyquist-plot domain for robust stability analysis of a single-input single-output (SISO) feedback system in the presence of uncertain gains, phases, and parameters subject to perturbations within prescribed bounds. Zames (1966a, 1966b) uses functional methods for robust and absolute stability analyses of input–output problems where two elements of the system representation are feedback interconnected. He finds conditions on each element in a so-called conic relation stability theorem that ensure the overall loop will remain stable when they are feedback interconnected. Safonov and Athans (1977) and Safonov (1980) extend Zames' conic relation stability theorem to multi-input multi-output (MIMO) systems and characterize sets of feedback laws in terms of sets of possible plant dynamics. Both of Zames' and Safonov's approaches are based on the concept of topological separation. They show that the feedback interconnection of two element H_1 and H_2 is robustly stable if and only if the graph of H_1 and the inverse graph of H_2 are topologically separated for any variations of H_1 and/or H_2 within the prescribed bounds. The main goal in this concept of robust stability is to find a separator or a set of separators that characterizes the input–output relation (i.e., graph) of either of the system elements H_1 or H_2. Similar approaches have been investigated and further developed by many researchers including Goh and Safonov (1995) (IQC separator), Rantzer and Megretski (1994) and Megretski and Rantzer (1997) (IQC theory), Scherer (1997, 2001) (Full-block S-procedure), and Iwasaki and Hara (1998) (Quadratic separator), to name a few. While they usually consider more general classes of uncertainties such as time-varying nonlinear functions, topological separation based robust stability analysis can be used to compute the upper-bounds on μ-calculations.

In addition to topological approaches to robustness analysis, an algebraic approach is found by Kharitonov (1978) and Bialas and Garloff (1985). They propose a stability criterion for uncertain polynomials in which the coefficients are independently varying in certain bounded intervals. This approach, however, is not applicable to general robust stability analysis, since most cases

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[1] Robustness margin refers to robust stability or performance of a system.

0098-1354/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.compchemeng.2013.09.018

of robustness problems have the coefficients of the characteristic polynomial that are interdependent. The robustness problems based on Lyapunov's method of stability have a substantial history, but they are beyond the scope of this paper and we, instead, refer the readers to research monographs in nonlinear system theory (Khalil, 2002; Vidyasagar, 1993).

Apart from reviewing historical development of robust analysis in control theory, the principal purpose of this paper is to provide a comprehensive overview of research related to computational complexity of μ-calculation problems and some of their descendants. As the structured singular value provides a way to measure the exact robustness of many classes of structured uncertain systems, there have been numerous research efforts to develop efficient algorithms for computing μ. Several researchers independently show that the exact μ-calculations for systems with real (Braatz, Young, Doyle, & Morari, 1994; Coxson & DeMarco, 1994; Nemirovskii, 1993; Poljak & Rohn, 1993), mixed (Braatz et al., 1994), and complex (Toker & Özbay, 1998) uncertainties are NP-hard problems. After the exact μ-calculations are proven to be NP-hard, it is shown that arbitrarily close approximations to the exact μ are also NP-hard problems for systems with real (Fu, 1997) and mixed (Braatz & Russell, 1999) uncertainties. Conservatism of some μ upper bounds is investigated by many researchers (Megretski, 1993a, 1993b; Megretski & Treil, 1993; Rump, 2001; Treil, 2000). Probabilistic randomized algorithms (Khargonekar & Tikku, 1996; Stengel & Ray, 1991; Tempo, Calafiore, & Dabbene, 2005; Vidyasagar, 1997, 1998; Vidyasagar & Blondel, 2001) to compute robustness margin are alternative approaches in which uncertainties are considered as random variables and robustness is evaluated in the probabilistic sense. The volume ratio of the set of destabilizing uncertainties to the whole compact uncertainty set is analytically or approximately computed with respect to a specific probability measure. In particular, sampling-based approximation approaches provide certain levels of accuracy and confidence in the obtained answers to robust stability and performance, which depend on the number of scenarios simulated. Polynomial-time dimension reduction methods (Beck, Doyle, & Glover, 1996; Russell & Braatz, 1998; Russell, Power, & Braatz, 1997) are also used to reduce the computational burden of calculating μ for large-scale systems.

In addition to robustness margin computation problems, there have been also many research efforts to study computational complexity of problems related to systems and control theory. Blondel and Tsitsiklis (2000) provide a complete survey of computational complexity results in system and control up to 2000. In particular, many important control problems can be formulated as bilinear matrix inequalities (BMIs) (see VanAntwerp & Braatz, 2000, for a review of such problems) and the problem of checking solvability of a BMI is NP-hard (Toker & Ozbay, 1995). *DK*-iteration for controller design using an upper-bound of μ is equivalent to finding a local solution for a BMI problem via alternatively freezing decision variables at each iteration step.

This paper is organized as follows. We first provide, in Section 2, a concise, but complete introduction to computational complexity theory that is able to characterize the inherent difficulty of checking solvability or calculating a solution for a problem under study. Several NP-complete and NP-hard problems that are used to study computational complexity of μ-calculation are presented. Section 3 overviews results on computational complexity of exact μ-calculation problems and shows the theoretical developments

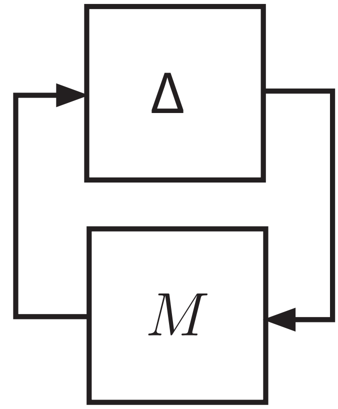

Fig. 1. Feedback interconnected system.

of research on computational complexity of different classes of μ-calculations and the connections of existing results. Section 4 presents results on the study of computational complexity of approximate μ-calculation problems. A correct definition for ϵ-approximation problems is given and several available results on NP-hardness of ϵ-approximation of μ problems are presented. Section 5 studies the conservatism of robust stability margin computation using μ upper-bound which is a convex problem. Some results showing that approximate upper-bounds can be arbitrarily conservative are presented. Followed by NP-hardness of μ computation and conservatism of convex upper-bounds that all consider deterministic worst-cases, alternative approaches to assess robustness of a system are investigated. In Section 6, polynomial-time model reduction methods and probabilistic randomized algorithms to compute robustness margins are reviewed, among many alternative approaches of μ computation. Section 7 concludes this paper. For easy reference, the results are displayed in the form of theorems, while most of the presented results are extracted from existing works.

Notation. Define the set of matrices of block-diagonal perturbations given by

$$\boldsymbol{\Delta} \triangleq \Big\{ \mathrm{diag}(\delta_1^r \mathbf{I}_{r_1}, \ldots, \delta_k^r \mathbf{I}_{r_k}, \delta_{k+1}^c \mathbf{I}_{r_{k+1}}, \ldots, \delta_m^c \mathbf{I}_{r_m}, \Delta_{r_{m+1}}, \ldots, \Delta_{r_{m_C}}) : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \ \Delta_i \in \mathbb{C}^{r_i \times r_i}, \sum r_i = \ell \Big\} \quad (1)$$

and $\mathbf{B\Delta}$ is the set of unity norm-bounded perturbations from $\boldsymbol{\Delta}$.

For a given matrix $M \in \mathbb{C}^{m\times m}$, the structured singular value (Doyle, 1982; Fan, Tits, & Doyle, 1991) is defined as

$$\mu_{\boldsymbol{\Delta}}(M) \triangleq \begin{cases} 0 & \text{if there exists no } \Delta \in \boldsymbol{\Delta} \text{ such that } \det(\mathbf{I} - M\Delta) = 0 \\ (\min\limits_{\Delta \in \boldsymbol{\Delta}}\{\overline{\sigma}(\Delta) : \det(\mathbf{I} - M\Delta) = 0\})^{-1} & \text{otherwise} \end{cases} \quad (2)$$

in which more general classes of structured uncertainties can be handled without introducing any further conservatism. Without loss of generality we have taken M and each subblock of Δ to be square. In this context, $\mu_{\boldsymbol{\Delta}}(M)$ defines a measure of the smallest structured $\Delta^\star \in \boldsymbol{\Delta}$ that destabilizes of the feedback interconnected system depicted in Fig. 1 and the norm of this destabilizing uncertainty $\Delta^\star$ is quantified as $(\mu_{\boldsymbol{\Delta}}(M))^{-1}$.

The set of real numbers is $\mathbb{R}$; the set of complex numbers is $\mathbb{C}$, and the set of rationale is $\mathbb{Q}$. A^{T} and A^{*} refer to the transpose and the conjugate transpose of A, respectively. The Euclidean 2-norm of vector x is defined by $\|x\|_2 = \sqrt{x^{\mathrm{T}}x}$ and the vector ∞-norm of x is defined by $\|x\|_\infty = \max\limits_i |x_i|$. For a vector $x \in \mathbb{C}^n$, $|x| = [|x_1|, \ldots, |x_n|]^{\mathrm{T}}$ denotes the vector of absolute values of its entries. The maximum singular value of matrix A is represented by $\overline{\sigma}(A)$, and $\mathbf{I}$ is the identity matrix of compatible dimension. An $\mathbf{0}$ will be used