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Solving linear and quadratic programs with an analog circuit

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ABSTRACT

We present the design of an analog circuit which solves linear programming (LP) or quadratic programming (QP) problem. In particular, the steady-state circuit voltages are the components of the LP (QP) optimal solution. The paper shows how to construct the circuit and provides a proof of equivalence between the circuit and the LP (QP) problem. The proposed method is used to implement an LP-based Model Predictive Controller by using an analog circuit. Simulative and experimental results show the effectiveness of the proposed approach.

1. Introduction

In 2002, Bemporad, Morari, Dua, and Pistikopoulos showed how to compute the solution to constrained finite-time optimal control problems for discrete-time linear systems as a piecewise affine state-feedback law (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Bemporad, Borrelli, & Morari, 2002b). Such a law is computed off-line by using a multi-parametric programming solver which divides the state space into polyhedral regions, and for each region determines the linear gain and offset which produces the optimal control action. This state-feedback law is often referred to as the "explicit solution". Since many control problems belong to this class, either in their natural form or after an approximation and abstraction step, their solution has been studied for decades. However, until that work, as there was no knowledge about the functional form and structure of closed form solutions, computations resorted to some approximation such as gridding or functional interpolation.

Enlightened by that breakthrough, Morari's research group started developing a new theory for optimal control of discrete-time linear systems, constrained linear systems, and hybrid systems. The theory (1) unveils the existence and the properties of the closed form solutions (Bemporad, Borrelli, & Morari, 2002a, 2003; Borrelli, 2003; Borrelli, Baoti, Bemporad, & Morari, 2005; Grieder, Borrelli, Torrisi, & Morari, 2004; Maeder, Borrelli, & Morari, 2009; Morari, Baotic, & Borrelli, 2003), (2) explains the effect of uncertainties on the control of constrained systems (Bemporad et al., 2003; Borrelli, 2003), (3) shows how to use linear and nonlinear multiparametric-programming to compute the closed forms solutions (Bageshwar & Borrelli, 2009; Bemporad et al., 2002a; Borrelli, Bemporad, & Morari, 2003) and (4) sheds light on the tight link between the desired optimality and the robustness of closed-loop systems and what can actually be achieved on resource-constrained embedded control hardware (in terms of CPU and storage) (Borrelli, Baoti, Pekar, & Stewart, 2010; Borrelli, Falcone, Pekar, & Stewart, 2009). The theory also simplifies and unifies much of the previous work for special classes of systems. In particular, it reduces to the well known Linear Quadratic Regulator for unconstrained linear systems. As an example, now we know the answer to the question "What is the solution to an LQR problem if the system states and inputs are constrained?". In Borrelli et al. (2005) it was shown that the state feedback control law is continuous and piecewise affine and that the value function is convex and continuously differentiable. For hybrid systems, it was also shown that the optimal control law is, in general, piecewise affine over non-convex and disconnected sets. The class of hybrid systems for which these results apply is very large including systems with both internal and/or controllable switches (Borrelli et al., 2010).

These results have had important consequences for the implementation of Model Predictive Control (MPC) laws. Pre-computing offline the explicit piecewise affine feedback policy reduces the on-line computation for the receding horizon control law to a function evaluation, therefore avoiding the on-line solution of a mathematical program as it is done in Model Predictive Control. This research has enlarged in a very significant way the scope

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of applicability of Model Predictive Control to small-size/fast-sampled applications (Avni, Borrelli, Katzir, Rivlin, & Rotstein, 2006; Borrelli, 2003; Falcone, Borrelli, Asgari, Tseng, & Hrovat, 2007). Since then, Prof. Morari's group and his collaborators have continued to push the capabilities of MPC to faster processes. Recently, using the capabilities of field programmable gate array (FPGA) they have reached sampling times below five microseconds for problems with tens to a few hundreds of variables (Jerez et al., 2013; Jerez, Ling, Constantinides, & Kerrigan, 2012, 2012; Mariéthoz, Mäder, & Morari, 2009).

To honor this fundamental work, we have chosen to dedicate our original contribution to Professor Morari. In this paper we prove that Model Predictive Control can be implemented by using a simple analog circuit. We hope that this discovery will significantly enlarge the scope of applicability of Model Predictive Control. In fact, the proposed approach and technology could enable the real-time implementation of MPC controllers on the order of nanoseconds with very small power consumption if a VLSI (Very Large Scale Integrated) circuit technology is used.

Analog circuits for solving optimization problems have been extensively studied in the past (Dennis, 1959; Kennedy & Chua, 1988; Tank & Hopfield, 1986). Our renewed interests stem from MPC (Garcia, Prett, & Morari, 1989; Mayne, Rawlings, Rao, & Scokaert, 2000). In MPC at each sampling time, starting at the current state, an open-loop optimal control problem is solved over a finite horizon. The optimal command signal is applied to the process only during the following sampling interval. At the next time step, a new optimal control problem based on new measurements of the state is solved over a shifted horizon. The optimal solution relies on a dynamic model of the process, respects input and output constraints, and minimizes a performance index. When the model is linear and the performance index is based on two-norm, one-norm or ∞-norm, the resulting optimization problem can be cast as a linear program (LP) or a quadratic program (QP), where the state enters the right hand side (rhs) of the constraints.

We present the design of an analog circuit whose steady state voltages are the LP/QP optimizers. Thevenin's Theorem is used to prove that the proposed design yields a passive circuit. Passivity and KKT conditions of a tailored Quadratic Program are used to prove that the analog circuit solves the associated LP or QP. The proposed analog circuit can be used to repeatedly solve LPs or QPs with varying rhs and therefore it is suited for a linear MPC controller implementation. For some classes of applications the suggested implementation can be faster, cheaper and consume less power than digital implementation. A comparison to existing literature reveals that the proposed circuit is simpler and faster than previously published designs.

The paper is organized as follows. Existing literature is discussed in Section 2. We show how to construct an analog circuit from a given LP in Section 3. Section 4 proves the equivalence between the LP and the circuit. Section 6 shows how to extend the LP results to solve QP problem. Simulative and experimental results show the effectiveness of the approach in Section 7. Concluding remarks are presented in Section 8.

2. Previous work on analog optimization

2.1. Optimization problems and electrical networks

Consider the linear programming (LP) problem

$$\min_{V=[V_1,\ldots,V_n]^T} c^T V \tag{1a}$$

$$\text{s.t. } A_{\text{eq}} V = b_{\text{eq}} \tag{1b}$$

$$A_{\text{ineq}} V \le b_{\text{ineq}}, \tag{1c}$$

where $[V_1, \ldots, V_n]$ are the optimization variables, A_{ineq} and A_{eq} are matrices, and c, b_{eq} and b_{ineq} are column vectors. The equality and inequality operators are element-wise operators.

The monograph by Dennis (1959) presents an analog electrical network for solving an LP (1). In Dennis's work, the primal and dual optimization variables are represented by the circuit currents and voltages, respectively. A basic version of Dennis's circuit consists of resistors, current sources, voltage sources, and diodes. In this, circuit each element value of matrices A_{ineq} and A_{eq} is equal to the number of wires that are connected to a common node. Therefore, this circuit is limited to problems where the matrices A_{ineq} and A_{eq} contain only small integer values. An extended version of the circuit includes a multiport DC–DC transformer and can represent arbitrary matrices A_{ineq} and A_{eq}. Current distribution laws in electrical networks (also known as minimum dissipation of energy principle or Kirchoff's laws) are used to prove that the circuit converges to the solution of the optimization problem. This work had limited practical impact due to difficulties in implementing the circuit, and especially in implementing the multiport DC–DC transformer.

In later work, Chua, Lin, and Lum (1982) showed a different and more practical way to realize the multiport DC–DC transformer using operational amplifiers. In subsequent works, Chua (Chua & Lin, 1984;Kennedy & Chua, 1988) and Hopfield (Tank & Hopfield, 1986) proposed circuits to solve non-linear optimization problems of the form

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{s.t.} \quad & g_j(x) \le 0, \quad j = 1, \ldots, m, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the vector of optimization variables, $f(x)$ is the cost function, and $g_j(x)$ are the m constraint functions. The LP (1) was solved as a special case of problem (2) (Kennedy & Chua, 1988; Tank & Hopfield, 1986). The circuits proposed by Chua, Hopfield, and coauthors model the Karush–Kuhn–Tucker (KKT) conditions by representing primal variables as capacitor voltages and dual variables as currents. The dual variables are driven by the inequality constraint violations using high gain amplifiers. The circuit capacitors are charged with a current proportional to the gradient of the Lagrangian of problem (2)

$$\frac{\partial x_i}{\partial t} = -\left[\frac{\partial f(x)}{\partial x_i} + \sum_{j=1}^{m} I_j \frac{\partial g_j(x)}{\partial x_i}\right], \tag{3}$$

where $\partial x_i/\partial t$ is the capacitor voltage derivative and I_j is the current corresponding to the jth dual variable. The derivatives $\partial f/\partial x_i$ and $\partial g_j/\partial x_i$ are implemented by using combinations of analog electrical devices (Jackson, 1960). When the circuit reaches an equilibrium, the capacitor charge is constant ($\partial x_i/\partial t = 0$) and Eq. (3) becomes one of the KKT conditions. The authors prove that their circuit always reaches an equilibrium point that satisfies the KKT conditions. This is an elegant approach since the circuit can be intuitively mapped to the KKT equations. However, the time required for the capacitors to reach an equilibrium is non-negligible. This might be the reason for the relatively large settling time reported to be "tens of milliseconds" for those circuits in Kennedy and Chua (1988).

2.2. Applying analog circuits to MPC problems

The analog computing era declined before the widespread use of Model Predictive Control. Quero, Camacho, and Franquelo (1993) have been the first to study the implementation of analog MPC. They use the Hopfield circuit proposed in Tank and Hopfield (1986) to implement an MPC controller. The approach they propose is validated with an experimental circuit which reaches the equilibrium after a transient of 1.8 ms.