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Parametric optimization with uncertainty on the left hand side of linear programs

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ABSTRACT

Although parametric optimization with uncertainties on the objective function (OF) or on the so-called "right-hand-side" (RHS) of the constraints has been addressed successfully in recent papers, very little work exists on the same with uncertainties on the left-hand-side (LHS) of the constraints or in the coefficients of the constraint matrix. The goal of this work has been to develop a systematic method to solve such parametric optimization problems. This is a very complex problem and we have begun with the simplest of optimization problems, namely the linear programming problem with a single parameter on the LHS. This study reviews the available work on parametric optimization, describes the challenges and issues specific to LHS parametric linear programming (LHS-pLP), and presents a solution algorithm using some classic results from matrix algebra.

1. Introduction

Generally, many problems of practical interest in the design and operation of chemical and bio-chemical processes are formulated as "deterministic optimization" problems in which the coefficients of the objective function and constraints are pre-specified constant parameters. Examples of such parameters include demands, availabilities, yields, prices, resource requirements, kinetic constants, transfer coefficients, etc. However, most of these parameters are usually functions of some other uncontrollable parameters that are seldom known with absolute certainty. These uncertainties can be categorized in three levels in the context of process system engineering, i.e. input/output, process and decision levels (Pertsinidis, Grossmann, & Mcrae, 1998). Input/output uncertainties are in the form of feed/product prices, availability of feed materials or demands/orders. Process level uncertainties generally stem from transport phenomena of the process or dynamic and unsteady-state behaviors of the system. Decision level uncertainties are in the form of changes in the decision criteria or objectives over different technical and non-technical conditions.

Clearly, the solution of a practical optimization problem is not complete with the mere determination of the optimal solution. Each variation in the values of the data coefficients changes the problem which may, in turn, affect the optimal solution found earlier. In order to develop an overall strategy to meet the various contingencies, one must study how the optimal solution will be affected by changes in the various model parameters. Optimization under uncertainty refers to the branch of optimization that studies the impact of uncertainties in the data or the model. In cases where the uncertain coefficients have a probabilistic nature, the optimization problem is called *stochastic programming* and, in cases where uncertainty is in the form of continuous parameters, it is called *parametric programming* (Khalilpour & Karimi, 2007).

It is, however, worth mentioning that mathematical programming from the viewpoint of parametric variations has been addressed at two levels, namely sensitivity analysis and parametric programming. While both study the changes in the optimal solution with respect to model parameters, they differ in the extent of their variations. The former addresses small perturbations of the parameters, while the latter addresses the full range of parameter changes and thus provides a complete map of the optimal solution in the space of the varying parameters (Borrelli, Bemporad, & Morari, 2003; Gal, 1982).

Parametric optimization methodologies are important, because a parametric solution is the most complete and exact for a deterministic optimization problem (Pertsinidis et al., 1998). The optimization literature classifies problems as linear programming

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Nomenclature

Subscripts

ABP	approximate break point
B	basic
BP	break point
N	non-basic
i	iteration
j	parameter
k	parameter

Superscripts

L	lower limit
U	upper limit

Parameters

$\mathbf{A}$	matrix of deterministic constraint coefficients
$\mathbf{A}(\lambda)$	matrix of parametric constraint coefficients
$\mathbf{A}_\lambda$	parametric matrix
$\mathbf{b}$	vector of RHS constants
$\mathbf{b}(\lambda)$	vector of RHS constants with parameters
$\mathbf{c}$	vector of objective function constant multipliers
$\mathbf{c}(\lambda)$	vector of objective function multipliers with parameters
$\mathbf{g}$	vector of inequality equations
$\mathbf{h}$	vector of constants
$\mathbf{I}$	identity matrix
$\mathbf{U}, \mathbf{C}, \mathbf{V}$	matrices of Woodbury equation
$\mathbf{u}, \mathbf{v}$	vectors of Sherman–Morrison equation

Variables: Binary

$\mathbf{y}$	vector of 0–1 binary variables

Continuous

$\mathbf{A}_B(\lambda)$, $\mathbf{A}_N(\lambda)$	matrix of parametric constraint for basic and non-basic variables
$\mathbf{A}_B$, $\mathbf{A}_N$	corresponding basic and non-basic matrices
$\boldsymbol{B}$	basic variable set
$\mathbf{c}_B$, $\mathbf{c}_N$	vector of objective function constant multipliers for basic and non-basic variables
$\boldsymbol{E}, \boldsymbol{F}$	parametric matrix for basic and non-basic variables
f	scalar objective function
$\boldsymbol{N}$	nonbasic variable set
$\mathbf{x}$	vector of continuous variables
$\mathbf{x}_B$, $\mathbf{x}_N$	sub-vector of $\boldsymbol{x}$ for basic and non-basic variables
$\bar{y}$	vector of relaxed variables
α_j, α_j^*	eigenvalues of constraint matrices
$[\lambda^L, \lambda^U]$	parameter range
$\boldsymbol{\rho}_i$	set of basic variables for i
λ	parameter
$\boldsymbol{\lambda}$	vector of parameters
λ_j	parameter (result of first prerequisite condition)
λ_k	parameter (result of second prerequisite condition)
λ^*	parameter (result of third prerequisite condition)
$\lambda_{ABP,i}$	approximate break point at iteration i
$\lambda_{BP,i}$	break point at iteration i
$\Delta\lambda$	step-size value

(LP), nonlinear programming (NLP), integer programming (IP), mixed integer programming (MIP), etc., depending on the nature of the decision variables, objectives and constraints. There has been extensive research to solve all these problems in a parametric manner. The next few sections provide an overview of the literature on some parametric optimization problems.

2. Parametric linear programming (pLP)

A deterministic LP can be written in the following form.

$$f = \min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge 0\} \tag{1}$$

where $\boldsymbol{c}$ and $\boldsymbol{b}$ are vectors of constants, $\boldsymbol{A}$ is the matrix of constraint coefficients and $\boldsymbol{x}$ is the vector of variables. The parametric linear program (pLP) corresponding to the above LP is given by,

$$f(\lambda) = \min\{\boldsymbol{c}(\lambda)^T\boldsymbol{x} : \boldsymbol{A}(\lambda)\boldsymbol{x} \le \boldsymbol{b}(\lambda), \boldsymbol{x} \ge 0\} \tag{2}$$

where λ is the subset Λ of parametric space, $\boldsymbol{A}(\lambda)$, $\boldsymbol{b}(\lambda)$, and $\boldsymbol{c}(\lambda)$ are functions of λ.

The literature on pLP (and other forms of optimization problems) is generally categorized into three groups based on the locations of uncertain parameters.

Right-hand-side pLP (RHS-pLP)

When the uncertainty is on the right-hand-side (RHS) of the constraints, i.e. vector $\boldsymbol{b}$, then the problem is called RHS-pLP.

$$f(\lambda) = \min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}(\lambda), \boldsymbol{x} \ge 0\} \tag{3}$$

Here, $\boldsymbol{b}(\lambda) = \boldsymbol{b} + \boldsymbol{h}\lambda$ is a linear function where $\boldsymbol{h}$ is the vector of constants, and

Objective function pLP (OF-pLP)

When the uncertainty is in the objective function (OF), i.e. vector $\boldsymbol{c}$, then the problem is called OF-pLP.

$$f(\lambda) = \min\{\boldsymbol{c}(\lambda)^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge 0\} \tag{4}$$

Here, $\boldsymbol{c}(\lambda) = \boldsymbol{c} + \boldsymbol{d}\lambda$ is the linear perturbation and $\boldsymbol{d}$ is the vector of constants.

Note that OF-pLP can be transformed into an RHS-pLP, and vice versa, so they can be easily transformed to their primal or dual forms.

Left-hand-side pLP (LHS-pLP)

When the uncertainty is in the constraint coefficient matrix, i.e. matrix $\boldsymbol{A}$, then the problem is called LHS-pLP. This problem is denoted as:

$$f(\lambda) = \min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}(\lambda)\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge 0\} \tag{5}$$

where $\boldsymbol{A}(\lambda) = \boldsymbol{A} + \boldsymbol{A}_\lambda$ is a parameterized set of the $m \times n$ matrix with $\boldsymbol{A}$ as the matrix of deterministic constraint coefficients. $\boldsymbol{A}_\lambda$ is a matrix of $m \times n$ size containing the parameters $\lambda_{ij} \in \Lambda \subseteq \Re^{m,n}$. In cases where only one parameter exists, $\boldsymbol{A}(\lambda) = \boldsymbol{A} + \boldsymbol{A}_0\lambda$ where $\boldsymbol{A}_0$ is a matrix of the parameter coefficient and λ is the one-dimensional parameter $\lambda \in \Lambda \subseteq \Re$.

2.1. Literature on pLP

The term "parametric optimization" was first introduced by Manne in his 1953 paper, about six years after Dantzig developed his Simplex algorithm in 1947. In the following years 1954 and 1955, Saaty and Gass published three papers on OF-pLP (Gass & Saaty, 1955; Saaty & Gass, 1954b, 1954a). These papers were the first publications addressing parametric programming with uncertainty on the objective function. Immediately after the introduction of these methods, the first applications to solve certain kinds of mathematical programming problems appeared (Gal, 1980).

It seems that during the following ten years (1955–1965), the development of new parametric programming solutions did not receive much attention. Rather, most of the efforts in the optimization field were focused on applications (Charnes & Cooper, 1961) or the further development of continuous LP to mixed integer problems (Benders, 1962; Gomory, 1958). Undoubtedly, the most important work in the parametric programming field is the paper by Gal and Nedoma (1972) which presented a systematic method