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On the linear stability of one- and two-layer Boussinesq-type equations for wave propagation over uneven beds

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ABSTRACT

Bousssinesq-type equations are a powerful tool to model the wave propagation from intermediate waters to the shore. By construction, these equations have a good performance in weakly dispersive conditions, and a great effort has been done during the last 20 years to increase their range of application to deeper waters; the improved equations introduce free coefficients that are chosen for this purpose. Some of the improved sets of equations show instabilities when numerically solved over uneven beds. In this work we show how these instabilities can be due to the equations (including the values of the involved coefficients) and not to the numerical scheme. We further introduce new sets of coefficients that optimize the linear performance while improving the linear stability of the equations.

1. Introduction

As water waves travel to the coast, the water depth, h, is reduced from hundreds or thousands of meters in the open sea to a few meters, and eventually zero, at the shore. In the open sea the wave height, H, is much smaller than the water depth, so that $H/h \ll 1$. Conversely, in the nearshore region the wavelength, λ, is usually much larger than the water depth, *i.e.* $\lambda/h \gg 1$ (or $kh \ll 1$, with $k \equiv 2\pi/\lambda$ being the wavenumber). Now the horizontal velocity profile is nearly uniform in the water column and the wave celerity is independent of the wave period, so that there is no frequency *dispersion*. The very well-known Shallow Water Equations (SWEs) apply in this region.

There is an intermediate zone where $H/h \lll 1$ and $kh \lll 1$. The Boussinesq Equations (BEs) were developed to represent water wave propagation in this region. BEs can be seen as an extension of the SWEs that includes dispersion in a perturbative way. BEs by Peregrine (1967) were obtained for weakly dispersive and weakly non-linear conditions. The extension to weakly dispersive but arbitrary (or "*fully*") non-linear conditions are very popular nowadays (Green and Naghdi, 1976; Wei and Kirby, 1995; Madsen and Schaffer, 1998; do Carmo, 2013), and are usually referred to as Serre's Equations, after Serre (1953), or also as Boussinesq *type* Equations (BTEs hereafter).

BEs and BTEs have ensured a good performance under weakly dispersive conditions (BEs for weakly non-linear conditions and BTEs for arbitrary non-linear conditions), by construction. In order to assess the performance under stronger dispersive conditions, BEs and BTEs are linearized and compared to linear and fully dispersive theories such as Airy and Mild Slope Equations (Dean and Dalrymple, 1984). The comparison is made in terms of wave celerity (*linear dispersion*) and wave shoaling over mild slopes (*linear shoaling*). The weakly non-linear performance is, also, usually compared to the second order Stokes theory for flat beds (Schaffer, 1996). Since BEs and their corresponding BTEs are identical in their linear weakly dispersive terms, the comparisons give the same results using the BEs or their corresponding fully non-linear extensions (BTEs).

Much of the research in this area during the last 20 years has been devoted to improve the linear properties of the equations. Leaving aside higher order (in dispersive terms) equations (Gobbi et al., 2000), which include spatial derivatives of order five, three main different approaches can be distinguished to this end: (i) Madsen and Sorensen (1992) proposed an *enhancement* technique so as to introduce new terms that improve the dispersive performance, and Beji and Nadaoka (1996) proposed an alternative set of enhanced equations; (ii) Nwogu (1993) introduced a new set of BEs written for the velocity at $z_\alpha = \alpha h$ (instead of the depth averaged velocity), and chose $\alpha = -0.53096$ to improve the linear dispersion up to $kh \sim 3$; the corresponding BTEs (fully non-linear extension) were introduced by Wei et al. (1995); and (iii) Lynett and Liu (2004) proposed a multilayer approach (previous equations were one-layer). The above three techniques have been

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further combined: Madsen and Schaffer (1998) combined the first two approaches, and Simarro et al. (2013) all three approaches. Actually, Lynett and Liu (2004) already used the approach by Nwogu (1993) within each layer. All the above BTEs include free coefficients which are chosen so as to mimic the linear and fully dispersive theory. In general, the more coefficients the better performance.

When applied to certain bathymetries (particularly when they have steep slopes), the numerical solution of the above BTEs may show instability problems. These instabilities may arise from the equations themselves (depending on choice for the free coefficients), and not from the numerical scheme employed to solve them. For instance, for the simplest flat bed case, the coefficient α of Nwogu must be (this is shown in Section 3)

$$-1.58 \lesssim \alpha \lesssim -0.42, \quad (1)$$

for the equations to be linearly stable. Nwogu (1993) proposed $\alpha = -0.53096$, which falls within the above range. Similar conditions are already known for other sets of equations, for the flat bed case. These conditions do not ensure, however, the stability of the equations for uneven bathymetries and, to the authors knowledge, no further research has been done in this regard.

The goal of this work is to obtain the sets of coefficients that optimize the linear properties of a wide range of BTEs while improving the linear stability over uneven bathymetries.

The work is structured as follows: Section 2 introduces the sets of BTEs considered and shows how to assess the linear dispersion and linear shoaling errors in a new simple way; Section 3 introduces the problem of the linear stability for these equations and presents the strategy followed in Section 4 to obtain the convenient coefficients. Section 5 shows the applications in the numerical solution of the equations.

2. Governing equations

We consider one- and two-layer BTEs in this study. The equations under consideration are fully described in Appendix A. In this section we describe the equations only in terms of the free coefficients that they introduce as well as in their linear properties.

2.1. One layer equations

The one-layer BTEs analyzed here are those presented by Beji and Nadaoka (1996) and by Simarro et al. (2013), shown in Appendix A.1. Other systems of weakly dispersive and fully nonlinear equations are not analyzed here, but the same treatment presented below is applicable.

The equations by Beji and Nadaoka (1996) were introduced as a simpler alternative to those by Madsen and Sorensen (1992). They include one free parameter, β, and have shown to be particularly well conditioned to represent linear dispersion and shoaling (Simarro, 2013). Beji and Nadaoka (1996) first derived their equations for weakly non-linear conditions (BEs); in Appendix A.1 we introduce their fully non-linear extensions (BTEs).

The BTEs by Simarro et al. (2013) depart from the sets by Madsen and Schaffer (1998) and Galan et al. (2012) to further improve weakly non-linear and weakly dispersive performance, and include eight free coefficients: α, α_ϵ, δ, δ_ϵ, δ_h, γ, γ_ϵ and γ_h. The equations by Wei et al. (1995) –and hence Nwogu (1993)–, Madsen and Schaffer (1998), Kennedy et al. (2001) or Galan et al. (2012) are recovered as particular cases setting some of these coefficients null (Table 1).

The coefficients α_ϵ, δ_ϵ and γ_ϵ affect exclusively the non-linear performance of the above BTEs. This work focuses on linear aspects, which are independent of these three coefficients; for completeness, however, the values of α_ϵ, δ_ϵ and γ_ϵ will be provided following the strategy by Schaffer (1996). Also, in the linear case, the equations K01-1 (Kennedy et al., 2001), W95-1 (Wei et al., 1995) and N93-1 (Nwogu, 1993) reduce to the same equations, and here we will refer to them as N93-1 when dealing with linear aspects. Similarly, S13-1 (Simarro et al., 2013), G12-1 (Galan et al., 2012) and M98-1 (Madsen and Schaffer, 1998) coincide in the linear case, and we will refer to them as M98-1.

Table 1
Free coefficients for the one-layer BTEs (marked with "✓"; "–" means "does not apply"). B96-1: Beji and Nadaoka (1996); S13-1: one-layer Simarro et al. (2013); G12-1: Galan et al. (2012); M98-1: Madsen and Schaffer (1998); K01-1: Kennedy et al. (2001); W95-1: Wei et al. (1995); N93-1: Nwogu (1993).

Authors	β	α	α_ϵ	δ	δ_h	δ_ϵ	γ	γ_h	γ_ϵ
B96-1	✓	–	–	–	–	–	–	–	–
S13-1	–	✓	✓	✓	✓	✓	✓	✓	✓
G12-1	–	✓	0	✓	✓	✓	✓	✓	✓
M98-1	–	✓	0	✓	✓	0	✓	✓	0
K01-1	–	✓	✓	0	0	0	0	0	0
W95-1, N93-1	–	✓	0	0	0	0	0	0	0

Table 2
Free coefficients for the two-layer BTEs (marked with "✓"). L04-2 $\equiv$ Lynett and Liu (2004), S13-2 $\equiv$ two-layer Simarro et al. (2013).

Authors	α_1	$\alpha_{\epsilon,1}$	β_1	$\beta_{\epsilon,1}$	α_2	$\alpha_{\epsilon,2}$	δ	δ_h	δ_ϵ	γ	γ_h	γ_ϵ
S13-2	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
L04-2	✓	✓	✓	✓	✓	✓	0	0	0	0	0	0

For later use, we recall that an alternative to $\xi \equiv kh$ to express the range of validity of BTEs is $\kappa \equiv \omega^2 h/g$, with ω being the wave angular frequency and g the gravitational acceleration. For linear waves (Airy theory) $\kappa = \xi \tanh \xi$ so that $\kappa \sim \xi^2$ for $\xi \ll 1$ and $\kappa \sim \xi$ for $kh \gtrsim 3$. This number, κ, is proportional to the parameter h/λ_0 used by other authors (Madsen et al., 1991; Nwogu, 1993), where $\lambda_0 \equiv 2\pi g/\omega^2$ is the wavelength in deep waters.

2.2. Linear properties

Being $c = \omega/k$ the wave celerity, the dispersion equation of the one-layer BTEs is

$$\left\{\frac{c^2}{gh} = \right\} \frac{\kappa}{\xi^2} = \frac{1+\rho_1\xi^2}{1+\rho_2\xi^2}\frac{1+\rho_3\xi^2}{1+\rho_4\xi^2} \quad \{\equiv D(\xi)\}, \quad (2)$$

where the coefficients ρ_j depend on β for B96-1, on α for N93-1 and on α, δ and γ for M98-1. Some of the coefficients ρ_j can be null. For N93-1, *e.g.*

$$\rho_1 = \rho_2 = 0, \quad \rho_3 = -\frac{3\alpha^2+6\alpha+2}{6}, \quad \rho_4 = -\frac{\alpha^2+2\alpha}{2}. \quad (3)$$

The error in the linear wave celerity (*linear dispersion*) is defined as

$$\epsilon_c(\kappa) \equiv \frac{c}{c_A} - 1, \quad (4)$$

where c_A is the celerity obtained from the linear fully dispersive theory (Airy). The error is expressed as a function of κ for the same reasons argued by Galan et al. (2012).

It is known that $\partial\omega/\partial k$ obtained from the above dispersion equation does not satisfy $A^2 c_g = \text{constant}$, with A being the wave amplitude (*i.e.* Beji and Nadaoka, 1996; Schaffer and Madsen, 1998). The proper procedure to assess the linear shoaling of BTEs originally considered the so-called shoaling gradient (Madsen and Sorensen, 1992), and, in order to obtain the error for the wave